CONJUGATE LINE CONGRUENCES CONTAINED IN A BUNDLE OF QUADRIC SURFACES*

BY

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Introduction.

The cubic complex defined by the generators of both systems in a bundle of quadric surfaces has been investigated by MONTESANO,† and his methods employed by FANO‡ to determine a certain congruence of order three. A method, similar in part, had been previously found by STAHL,§ in connection with congruences of the second order whose lines can be arranged on a system of quadric surfaces.

The questions discussed in these papers suggest the more general one of conjugate congruences,‖ that is, those in which the two systems of generators of ∞^1 quadrics belonging to the bundle define two distinct congruences having the same focal surface. In the following paper I propose to derive such congruences and obtain a number of their properties. It will be seen that *the equations of such congruences of every order and of genus ranging from zero to a definite function of the order can be expressed rationally in terms of the eight points common to all the quadrics of the bundle.* The focal surface of each congruence remains invariant under a discontinuous birational transformation group of infinite order.

The preceding memoirs are all synthetic in their treatment; the present discussion is analytic, and the proofs of a few theorems already known will be given from a different point of view. Incidentally, extensive families of contact curves of a given quartic are obtained by algebraic methods.

* Presented to the Society, April 30, 1910.

† *Su di un complesso di rette di terzo grado*, Memorie di Bologna, ser. 5, vol. 3 (1893), pp. 549–577.

‡ *Nuove ricerche sulle congruenze di rette del 3° ordine*, Memorie di Torino, ser. 2, vol. 51 (1901), pp. 1–79.

§ *Ueber Strahlensysteme zweiter Ordnung*, Crelle's Journal, vol. 95 (1883), pp. 297–316.

‖ The idea of conjugate congruences has been developed from a different point of view by BALDUS: *Ueber Strahlensysteme, welche unendlich viele Regelflächen 2 Grades enthalten*, Erlangen dissertation, 1910. The only case to which his method and my own both apply is that of congruences of order two.

§ 1. *The involution I.*

1. Let $H_1(x)=0$, $H_2=0$, $H_3=0$ be the equations of three quadric surfaces, and let $B_i(i=1, \cdots, 8)$ be the eight points common to them. The bundle of quadrics

$$\Sigma = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 0 \tag{1}$$

defines a complex of order 3, the cone belonging to any point P being determined by the space quartic curve c_4 passing through P. This c_4 is the basis curve of the pencil of quadrics of (1) determined by P. The lines of the complex Σ can be put into (1, 1) correspondence with the points of space. Given any point $P \equiv (\xi_1, \xi_2, \xi_3, \xi_4)$, the line l joining P to B_1 will uniquely determine a quadric of Σ containing l. A generator m' of the other regulus passes through P. With P we may therefore associate the line m' of the complex. Conversely, any line m' of Σ will uniquely determine a quadric of the bundle. The quadric passes through B_1, through which passes a line l of the regulus to which m' does not belong. If l cuts m' in P, we may associate m' with the point P. When m' describes the whole regulus H determined by it, P will describe the line l; hence associated with every regulus is a line of the bundle B_1. The tangent plane to H at B_1 contains l and a line l' of the other regulus, so that there is associated with each quadric of Σ a pair of lines l, l' in the bundle B_1, each of which determines the other. These pairs determine a two-dimensional involution I.

The quadrics of Σ *are in* (1, 1) *correspondence with the pairs of lines of the involution* I *in the bundle* B_1.

2. Moreover, the quadrics of Σ are in (1, 1) correspondence with the points of the λ-plane. The pencil of quadrics in Σ passing through the point $P \equiv (\xi)$ corresponds to the straight line

$$\lambda_1 H_1(\xi) + \lambda_2 H_2(\xi) + \lambda_3 H_3(\xi) = 0.$$

The basis c_4 of the pencil of quadrics passes through B_1, hence the lines l, l' corresponding to quadrics of this pencil will describe a cubic cone K_3 passing through the other basis points B_i.

To the ∞^2 *straight lines of the* λ*-plane correspond the* ∞^2 *cubic cones containing the seven fixed lines* l_{1k} $(k=2, \cdots, 8)$.

Any two lines in λ intersect in a point; two cones K_3 intersect in two lines which are conjugate in I; this pair of lines is the image of the given point.

The relations between (λ) and (H_2) are obtained by solving for $\lambda_1 : \lambda_2 : \lambda_3$ the equations of a quadric H and its tangent plane at B_1. Let $B_1 \equiv (\eta_1, \eta_2, \eta_3, \eta_4)$, and write $\eta_i x_4 - \eta_4 x_i = y_i$. The equation of the tangent plane becomes $\Sigma \lambda_i v_i = 0$, and H is of the form $\Sigma \lambda_i V_i + (\Sigma \lambda_i v_i) v_4 = 0$, wherein v_i, V_i are certain homogeneous linear and quadratic functions respectively in y_1, y_2, y_3, and $v_4 = 0$ defines a plane not passing through B_1. Solving for λ_i we have

$$(2) \qquad \lambda_1 : \lambda_2 : \lambda_3 = [v_2 V_3 - v_3 V_2) : (v_3 V_1 - v_1 V_3) : (v_1 V_2 - v_2 V_1).$$

3. Any congruence contained in Σ can be defined by a relation among the λ_i, or by a curve $f_n(\lambda)$ in the λ-plane. Corresponding to any line m of this congruence is a point P, the locus of which is the image cone of the given curve f_n. It will be of order $3n$ and have each line l_{1k} for an n-fold line. The order of the congruence will be $2n$ and the two systems of reguli will in general be contained in it.

§ 2. *Conjugate Congruences.*

4. It may happen that l, l' describe two different cones. For this purpose it is necessary that K_{3n}, image of an irreducible $f_n(\lambda)$, breaks up into two cones which are transformed into each other by the involution I. Such particular curves can be obtained by starting with the cone. Let l describe the cone $K_n(v)$ with vertex at B_1. From the relations

$$(3) \qquad \Sigma v_i \lambda_i = 0, \qquad \Sigma V_i \lambda_i = 0, \qquad K_n(v_1, v_2, v_3) = 0$$

the v_i should be eliminated. The resultant in λ is the equation of the required curve. If K_n passes through B_i, all the equations will be satisfied by its coördinates. If l_{ik} is the multiplicity of the line joining B_i to B_k, then Σl_{1k} must be subtracted from the order of the resultant which is $n + 2n$.

The image of a cone of order n passing l_{1k} times through B_k is a curve in the λ-plane of order $3n - \Sigma l_{1k}$. The lines of the cone and the points of the curve are in (1, 1) correspondence. When l describes K_n, l' will describe another cone, birationally equivalent to it, and conjugate in the involution I. If the values of λ_i from (2) be substituted in the equation of the curve, K_n will be a factor. The other factor will give the equation of the other cone.

But the relations between l, l' can be determined directly. In the net of cubic cones $\Sigma a_i K_3^{(i)}$ a pencil is determined by one more basis line r. The cones of this pencil now contain eight fixed basis lines, hence they contain a ninth r'. If r_i be the coördinates of r, the relations between r, r' may be expressed by the equations

$$\frac{K_3^{(1)}(r')}{K_3^{(1)}(r)} = \frac{K_3^{(2)}(r')}{K_3^{(2)}(r)} = \frac{K_3^{(3)}(r')}{K_3\ (r)}.$$

When r describes a plane, r' will describe a rational cone of order 8, having the seven lines l_{1k} for three-fold edges. The involution I represents a Cremona transformation of order 8. If l describes a plane pencil passing through B_i, K' is of order 5 and has each of the remaining lines l_{1k} for double edge. If the plane passes through B_i, B_k, K' is a quadric cone through the other five.*

*This interesting transformation was first discussed by GEISER: *Ueber zwei geometrische Probleme*, Crelle's Journal, vol. 67 (1867), pp. 78–95; it has been further considered by BERTINI: *Ricerche sulle trasformazioni univoche involutorie nel piano*, Annali di Mate-

From the method in which it is produced it is evident that the Jacobian J_6 of the net, a sextic cone having each line l_{1k} for double edge, is invariant in I in such a way that every line will go into itself.

Every line in which $K(l)$ intersects J_6 must lie on $K'(l')$ also.

Since every line of J_6 goes into itself when operated upon by I, the corresponding quadric of Σ is a cone; J_6 is the projection from B_1 of the locus of the vertices of the cones contained in the bundle (1). Moreover, since the lines which K, K' have in common apart from those on J_6 occur in pairs, the image in the λ-plane is a double point on the image $f(\lambda)$ and the quadric H belongs to both the congruences σ, the locus of l, and σ', the locus of l'.

5. The two congruences σ, σ' have the same focal surface ϕ. Its order is twice the class of $f(\lambda)$. The points B_k are all singular upon ϕ, of an order half the order of the surface. The other isolated singular points are of order two; they are the vertices of the quadric cones belonging to the congruence and contained in Σ. Their number is the number of lines common to K and J_6 apart from the lines l_{1k}. The focal surface also contains a cuspidal curve and a double curve, but the points of these are not singular points for either congruence.

§ 3. *Double λ-Plane.*

6. A cone K whose points are the images of the lines of σ is a singular cone of σ', and conversely. It was seen that a $(1, 1)$ correspondence exists between the points of the λ-plane and the quadrics of the bundle Σ. Now consider the λ-plane as double, or two-sheeted, the same point defining the l or the l' generation of the image quadric according as it is regarded as belonging to the first or second sheet. For values of λ_i corresponding to a cone of Σ the two systems l, l' coincide, hence at each such point of the λ-plane the two sheets must coincide. The values of λ_i that make the discriminant $\Delta_4(\lambda)$ of Σ vanish define the cones. This is a general non-singular quartic curve in the λ-plane. When a locus $f(\lambda)$ crosses $\Delta(\lambda)$ the two sheets cannot be distinguished from each other, hence a necessary condition that $f(\lambda)$ may define two conjugate congruences is that it shall touch $\Delta(\lambda)$ at every point which they have in common.

In case $f(\lambda)$ is rational, this is also sufficient, since rational curves are unipartite. In such cases λ_i can be expressed rationally in terms of a parameter t, which when substituted in Σ will define a family of quadrics of the form

$$\sum_{m=1}^{n} t^m H_m = 0.$$

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matica, ser. 2, vol. 8 (1877), pp. 244–282. It has also received minor mention by CAPORALI in the Rendiconti di Napoli (1879), by DOEHLEMANN in Schlömilch's Zeitschrift, vol. 36 (1891), and by STEINMETZ in the American Journal of Mathematics, vol. 14 (1891), pp. 39–66.

The necessary and sufficient condition that a relation of genus zero between the coefficients λ_i of the quadrics of a bundle may define two conjugate congruences is that the discriminant, expressed in terms of the parameter, shall be a square.

For curves of genus greater than zero the condition is not sufficient on account of the existence of multipartite curves; one part may touch $\Delta(\lambda)$ in one region and another in another. However, the elimination of v_i from (3) will always define a contact curve of the proper kind. All the contact curves of $\Delta_4(\lambda)$ which belong to an odd characteristic can be obtained in this way.*

7. The equation of the focal surface is obtained by expressing the condition that the line $\Sigma H_i\lambda_i = 0$ shall touch the curve $f(\lambda) = 0$. If in the result $\phi(H_1, H_2, H_3) = 0$ we think of H_i as point coördinates, every quadric of the congruence is represented by a tangent to the curve ϕ, and the quartic curve along which the quadric touches the focal surface is the image of the point of contact. The residual points of intersection of the tangent and the curve have other quartic curves for images, hence:

Every quadric of a congruence contained in a linear bundle of quadrics touches the focal surface along a quartic curve and cuts it in a residual curve composed of space quartics of the first kind, all passing through the eight basis points of the bundle. Each of these residual quartics is itself the curve of contact of some quadric of the congruence.

Moreover, corresponding to every point of inflexion on $f(\lambda)$ there is a cusp on $\phi(H)$ and to every double tangent on f a double point on ϕ.

The focal surface has a cuspidal curve consisting of as many quartics as there are points of inflexion on $f(\lambda)$, and a double curve consisting of as many quartics as there are bitangents of $f(\lambda)$.

§ 4. *Problem of Projectivity.*

8. The points of a singular cone K with vertex at B_i uniquely define a congruence σ. The cone K' is found from K by the involution I, hence the conjugate congruence σ' is also defined. But each point B_i is the vertex of a singular cone, and any one of these would be sufficient to define the same congruence. These cones are thus all birationally equivalent to the given cone, and each must satisfy the relations analogous to (2) and (3). The double points on ϕ which are the images of lines of J_6 do not need to be considered, for they are vertices of quadric cones belonging to both σ and σ'.

Any two bundles B_i, B_k can be put into (1, 1) correspondence. A line m_i of B_i determines a quadric H of Σ. Since H passes through B_k there is just one line m_k through B_k lying on H and belonging to the same regulus as m_i.

*For the literature of contact curves of $\Delta_4(\lambda)$ see the Encyklopädie der mathematischen Wissenschaften, III C5, §§ 60–62, 67, 74.

The quadric H passes through the other basis points $B_1, \cdots, B_8$, through each of which passes one line belonging to the other regulus, hence:

The six planes containing any line of the bundle B_i and any six basis points respectively are projective with the six planes through the corresponding lines of the remaining bundle and the same six basis points.

Since the eighth basis-point is uniquely fixed when the other seven are known, the problem of finding m_k when m_i is given is reduced to the problem of projectivity which, stated for the plane, is as follows: Given any five points $A_1, \cdots, A_5$, any other set of five points $A_1', \cdots, A_5'$, and a point P, all in the same plane, to find a point P' such that the pencil $P'A_i'$ is projective with the pencil of five lines PA_i. This problem has been completely solved.*

The bundles B_i, B_k are equivalent under a Cremona transformation of order 5, having the remaining basis points for fundamental elements of the second order; to an arbitrary plane through B_i corresponds a rational quintic cone with vertex at B_k and having double edges passing through the other basis points. If the plane of B_i passes through B_1, the K_5 of B_1 is composite, consisting of a quadric cone through the other five edges and a cubic cone with a double edge passing through B_1 and passing once through each of the remaining basis points. If the pencil B_i passes through B_k, the K_3 of B_k passes through B_i. If the pencil B_i passes through B_1 and B_2, the proper image in B_k is the plane $B_kB_1B_2$. If K_n of B_i has a double or cuspidal edge not passing through any other basis point, its image in B_k will also have a double or cuspidal edge not passing through any other basis point, and similarly for its image in the involution I. The equations of this transformation can be obtained readily by three quadric inversions and two collineations, hence the complete system of singular cones of both σ and σ' can be expressed analytically when any one such cone of either is given.

§ 5. *Infinite Birational Groups.*

9. The focal surface ϕ is invariant under the operation of interchanging the points of contact of every line of σ. This transformation T_1 is birational and is of order 2. The congruence σ' defines another birational transformation T_2 of order 2. Except for particular cases arising from relations among the basis point B_i, the operation T_1T_2 is of infinite order.†

The equations of this transformation can be obtained directly by the same method as that previously employed in the case of rational conjugate congruences.‡

*STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 1 (1908), pp. 352–358. See also SCHOENFLIES: *Encyklopädie der math. Wissenschaften*, III AB5, § 16 (p. 442).

† SNYDER: *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15-24.

‡ SNYDER: *Surfaces invariant under infinite discontinuous birational groups defined by line congruences*, American Journal of Mathematics, vol. 32 (1910), pp. 177–185.

The fundamental points are the eight basis points and the vertices of the quadric cones belonging to both congruences; the fundamental curves are the curves of contact of these cones with the focal surface.

Among the quadrics of the congruence there are a finite number which touch the focal surface along two quartic curves. These double quadrics may arise in two different ways, and have in consequence different relations between the two quartic curves of contact.

If $K(v)$ has a double edge not passing through a basis point, $f(\lambda)$ will have a double point and $\phi(H)$ a corresponding double tangent. Let the images of the points of contact be c_4, c_4'. A line of σ will cut c_4 in P_1, P_2 and cut c_4' in P_1', P_2'. Corresponding to one sheet of $K(v)$ the first pair of points are foci of the given line, and to the other sheet, the second pair. Similarly, P_1 is focus for the σ' line passing through it, for the same sheet of $K(v)$.

Now suppose $K(v)$ intersects $K'(v)$ in a pair of conjugate lines l, l' in I. These two lines belong to the same quadric H, define a double point on $f(\lambda)$, and a double tangent to $\phi(H)$. Every line of both reguli of H belongs to both congruences σ and σ'. Let c_4, c_4' be the quartics of contact, and P a point on c_4. The lines m, m' both touch ϕ at P. As lines of σ, m has its foci on c_4, but m' on c_4'; as lines of σ', m has its foci on c_4', and m' on c_4. Every rational congruence of order > 2 contained in Σ must have a double quadric of one or the other of these forms. For orders higher than 3 both forms may appear in the same congruence.

The details of a few illustrative cases will now be discussed.

§ 6. *Images of Contact Curves.*

10. Let K_1 be a plane pencil with vertex at B_1 and passing through B_2, B_3. The image in λ is a bitangent to $\Delta_4(\lambda)$, hence the congruence is of order 1. Since $B_k (k = 4, 5, 6, 7, 8)$ is also the vertex of a pencil passing through B_2, B_3 it follows that the line B_2B_3 belongs to the σ' congruence, and all the lines of σ cut it. The pencil of quadrics in Σ which contains B_2B_3 has for its remaining basis curve a space cubic passing through the other six basis points and cutting B_2B_3 twice. The σ congruence therefore consists of the common secants of B_2B_3 and the associated cubic.

The images of the 28 bitangents of $\Delta_4(\lambda)$ *are the 28 lines* B_iB_k *and the 28 associated cubics passing through the remaining basis points.*

11. If K_1 passes through B_2 but not B_3, the λ image is a conic touching $\Delta_4(\lambda)$ in four points. The congruence is of order 2 and ϕ is of order 4. This is the well-known congruence of order 2, class 6, which has a singular plane. The plane K_1 can pass through any of the $8 \cdot 7/1 \cdot 2$ lines B_iB_k, and have ∞^1 positions in each case; each defines a contact conic of $\Delta_4(\lambda)$, hence:

The set of 28 series of ∞^1 contact conics of $\Delta_4(\lambda)$ are images of (2, 6) congruences having a singular plane.

12. A K_1 with vertex at B_1 but not containing any other basis points will cut J_6 in six lines. The image K_8' in I will cut K_1 in these six lines and also in one pair l, l' conjugate in I. The λ-image is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is rational, of order 3, class 9, and has one double quadric of the second kind. The focal surface is of order 8, has eight fourfold points and six double points. Its cuspidal curve is composed of three quartics passing through the eight fourfold points. The singular cones at the other basis points are rational K_5 having the lines joining all the others except B_1 for double edges. The conjugate congruence has B_1 for the vertex of a rational K_8 having triple edges passing through all the other basis points, and a K_4 from each of the remaining basis points with a triple edge passing through B_1 and passing simply through the remaining six. The vertices of the six quadric cones all lie in the given K_1.

13. A K_2 through five basis points will define the (1, 3) congruence already discussed. In fact the singular cones through B_2 and B_3 are exactly of this type.

A K_2 through four basis points defines the well known (2, 6) congruence without a basis plane.

The conjugate congruence is of the same form, hence:

The set of 35 series of ∞^1 contact conics of $\Delta_4(\lambda)$ are images of (2, 6) congruences having no singular plane.

If K_2 has its vertex at B_1 and passes through B_2, B_3, B_4, it will cut J_6 in six lines apart from lines l_{ik}. The conjugate K_7' will have l_{12}, l_{13}, l_{14} for double edges, and the remaining l_{ik} for triple edges. It will therefore cut K_2 in one pair of lines l, l', conjugate in I. The image in the λ plane is a nodal cubic touching $\Delta_4(\lambda)$ in six points. The congruence is of order 3, class 9, has a double quadric of the second kind, and contains six quadric cones. The focal surface is of order 8 and has many properties in common with the preceding case, but the vertices of the quadric cones do not lie in a plane, and the configuration of the singular cones at the basis points is different.

The points B_2, B_3, B_4 are vertices of cones of order 6, K_6 having the lines to the remaining two for triple edges and double edges passing through the four points B_5, B_6, B_7, B_8. Each passes simply through B_1. The conjugate congruence has each of these three points for vertex of a cubic cone having a double edge, passing simply through those four basis points, but not through the two remaining ones of the triad. The given congruence has B_5, B_6, B_7, B_8 as vertices of singular cones of order four; each has a double edge through B_2, B_3, B_4, and passes simply through the remaining points of the tetrad, but does not pass through B_1. The conjugate congruence has each of the points B_5,

B_6, B_7, B_8 for vertex of a quintic cone having the line to B_1 for triple edge: the remaining points of the tetrad are on double edges, and each cone passes simply through the triad B_2, B_3, B_4. It is now easy to obtain congruences of orders 4, 5, 6 and having three, six, or ten double quadrics of the second kind as images of a quadric cone having two, one, or no lines l_{1k}. No rational congruence of order higher than six can have a singular cone of order less than three.

14. A K_3 with six simple lines l_{1k} will define a congruence of order 3 without a double quadric. If B_2 is the basis point not lying on $K_3(B_1)$, then B_2 is the vertex of a similar cone, passing through the same six basis points, which are vertices of K_5, each having lines joining the other five for double edges. The conjugate congruence has sextic cones at B_1 and B_2, each having the line B_1B_2 for triple edge and each having double edges through the other six basis points. The points $B_3, \cdots, B_8$ are vertices of K_4's having double edges passing through B_1 and B_2, and passing simply through the other basis points.

These two conjugate congruences are the only ones without double quadrics that can be mapped upon elliptic cones. Since there are 28 pairs B_iB_k and each cone contains three undefined constants, these congruences are imaged in the λ-plane by 28 series of ∞^3 contact cubics of $\Delta_4(\lambda)$. The remaining 36 series do not define conjugate congruences. If the original $K_3(B_1)$ has a double edge not containing any other basis point, the resulting rational congruences will have a double quadric of the first kind. *We have thus found six rational congruences of order 3 contained in* Σ; these results are believed to be new.*

15. Another interesting illustration is furnished by the $K_6(B_1)$ having double edges passing through all the other basis points. The singular cones at the other basis points are of the same kind, and the conjugate congruence has an exactly similar configuration. The image in λ is a non-singular quartic curve touching $\Delta_4(\lambda)$ in eight points. The congruence is therefore of order four. *The focal surface is of order 24, has eight twelve-fold points and eight double points; it has a cuspidal curve of order 96 composed of 24 quartics of the first kind passing through the eight singular points, and a nodal curve of order 112 composed of 28 quartics of the same bundle.* As before, K_6 may have one or more double edges, giving rise to quartic congruences having one or more double quadrics of the first kind.

The two congruences of order 2, of genus zero, the two conjugate congruences of order 3, of genus one, and the self-conjugate congruence of order 4, of genus three, are the only possible conjugate congruences contained in Σ which do not have a double quadric.

Congruences which can be mapped birationally upon cones of given genus can

* The elliptic congruences of order three were found by FANO (l. c.); he does not mention congruences having a double quadric.

be constructed, but the corresponding order is not less than the minimum order of a plane curve of that genus.

16. Among the $K_6(B_1)$ having seven double edges through the other basis points are two exceptions, J_6 and the image of a general c_2 in the λ-plane. The former corresponds to $\Delta_4(\lambda)$ itself and is defined as the congruence formed by the cones in Σ. From the preceding theorem we see that its focal surface is of order 24, has 24 cuspidal quartics and 28 double quartics, but from No. 10 the latter consists of 28 lines l_{ik} and the 28 associated cubics; moreover, instead of having eight other double points, it has as double point every point of the sextic which is projected from B_1 by J_6. The other exceptional type of K_6 is defined by quadratic relations among the cubics forming the net. Every such cubic is an adjoint of the sextic and every pencil of such cubics having one basis line of K_6 will, from the nature of its equation, have the other on K_6 also, that is, K_6 is hyperelliptic.*

The $c_2(\lambda)$ cuts $\Delta_4(\lambda)$ in eight points, but the vertices of the corresponding cones are not singular points on ϕ_4. If U_i be linear functions of the H_k, the equation has the form

$$U_1 U_3 - U_2^2 = 0.$$

When the equation of a quartic surface can be written as a quadratic function of three quadrics, the tangent cone of order 6 from each associated double point is either composite or hyperelliptic.

This theorem is true for surfaces of any order defined by functions of U_1, U_2, U_3.

§ 7. *Particular Positions of Basis Points.*

17. Let it be supposed that four basis points B_1, B_2, B_3, B_4 lie in a plane π_1, and the other four in another plane π_1'. When P describes a plane pencil with vertex at B_1, its image in the involution I will be a cone K_7' of order 7, with l_{12}, l_{13}, l_{14} for double edges, and the lines from B_1 to the other basis points for triple lines. As the basis points appear unsymmetrically a singular cone of order n and vertex at B_k will be designated by $K_n(k\,;\, l_h, m_p, \cdots)$, $l_h \cdots$ indicating that the line joining B_k to B_l is an h-fold line, m_p that $B_k B_m$ is a p-fold line, etc. Moreover, the involutions I will be different for different points. This difference will be indicated by subscripts. The result of operating with I_1 upon $K_1(1)$ will be indicated by the symbol

$$K_1(1)\, I_1\, K_7'(1\,;\, 2_2, 3_2, 4_2, 5_3, 6_3, 7_3, 8_3).$$

The line $\pi_1 \pi_1'$ is now a part of the locus of vertices of quadric cones contained in the bundle Σ. The residual curve is a hyperelliptic quintic whose projecting cone from B_1 may be expressed by the symbol $J_5(1\,;\, 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_2)$.

*SNYDER: *On a special algebraic curve having a net of minimum adjoint curves*, Bulletin American Mathematical Society, vol. 14 (1907), pp. 70–74.

The discriminant curve $\Delta_4(\lambda)$ has one double point, the coördinates of which define the planes π_1, π_1' in Σ.

The 16 double tangents of $\Delta_4(\lambda)$ are images of the lines $B_k B_h$ ($k = 1, 2, 3, 4$; $h = 5, 6, 7, 8$) and the six tangents from the node are each the image of a pair of lines, one in π_1, one in π_1'. The transformation T_{ik}, defining the cone at B_k when that at B_i is given, is not changed from that of the general case when B_i, B_k both lie in π_1 or π_1' but is of order 4 when they lie in different planes. Thus we may write

$$K_1(1)\, T_{15} K_4(5; 2_2, 3_2, 4_2, 6_1, 7_1, 8_1).$$

The focal surface of the congruence of cones in Σ is now of order 20, contains 18 cuspidal quartics of the first kind, six double planes, six pairs of intersecting double lines, 16 other double lines, 16 double space cubics, and a double quintic.

The planes through B_1 and any basis point in π_1' define the general $(2, 5)$ congruences. The line of intersection of this plane and π_1 does not pass through any other basis point, hence it is the image of a cone breaking up into π_1 and π_1'. The cone consists of two plane pencils, one in π_1, the other in π_1', the vertices of each being on the line $\pi_1 \pi_1'$. In the conjugate congruence the planes of these pencils are interchanged.

The image conic in the λ-plane touches $\Delta_4(\lambda)$ in three points and passes through the node.

The singular cone $K_1(1; 2_1)$ defines a particular form of the $(2, 6)$ congruence having one singular plane.

A general $K_1(1)$ defines a $(3, 8)$ congruence, in which one cone breaks up into π_1, π_1'. The focal surface has one more double point and two more double planes than in the general case. The congruence contains one double quadric of the second kind.

An interesting example is the $(3, 7)$ congruence defined by the singular cone $K_2(1; 5_1, 6_1, 7_1)$. The complete configuration of singular cones of the two conjugate congruences is as follows:

$$K_2(1; 5_1, 6_1, 7_1), \qquad K_4(2; 3_1, 4_1, 5_2, 6_2, 7_2, 8_1),$$
$$K_5'(1; 2_1, 3_1, 4_1, 5_2, 6_2, 7_2, 8_3), \qquad K_3'(2; 1_1, 5_1, 6_1, 7_1, 8_2),$$
$$K_4(5; 1_1, 2_2, 3_2, 4_2, 6_1, 7_1),$$
$$K_3'(5; 1_1, 2_1, 3_1, 4_1, 8_2).$$

Those at B_3, B_4 can be obtained from those at B_2 by interchanging 2 and 3, 2 and 4 respectively, and those at B_6, B_7, B_8 by making similar changes in those at B_5.

The congruences contain a double quadric of the second kind consisting of the two pencils in π_1, π_1'. The image in the λ-plane consists of a cubic having a double point at the node of $\Delta_4(\lambda)$ and touching the latter in four points.

The cone $K_2(1;\ 4_1, 5_1, 6_1)$ defines a $(3, 8)$ congruence having five cones, one pair of planes, and a double quadric of the second kind.

The cone $K_2(1;\ 3_1, 4_1, 5_1)$ defines a $(3, 9)$ congruence containing six cones of Σ and a double quadric of the second kind.

The cubic cone $K_3(1;\ 3_1, 4_1, 5_1, 6_1, 7_1, 8_1)$ defines a $(3, 8)$ congruence having 5 quadric cones of Σ. If K_3 is rational the congruence contains a double quadric of the first kind.

The cone $K_3(1;\ 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)$ defines a $(3, 9)$ congruence as in the general case.

The preceding eighteen forms are the only possible types of congruences in Σ of order less than 4.

The non-hyperelliptic cone of order 6 having a double edge passing through each remaining basis point defines a $(4, 12)$ congruence as in the general case. The conjugate congruence is of the same type.

18. Let the basis points lie on two pairs of planes:

$$\pi_1 \equiv B_1B_2B_3B_4, \qquad \pi_1' \equiv B_5B_6B_7B_8;$$
$$\pi_2 \equiv B_1B_2B_5B_6, \qquad \pi_2' \equiv B_3B_4B_7B_8.$$

The two lines $\pi_1\pi_1'$ and $\pi_2\pi_2'$ and a space quartic of the first kind constitute the locus of the vertices of cones in Σ. The latter is projected from B_1 by the cone $J_4(1;\ 3_1, 4_1, 5_1, 6_1, 7_2, 8_2)$. The involution I transforms a plane pencil into a cone of order 6 of the form

$$K_1(1)IK_6'(1;\ 2_1, 3_2, 4_2, 5_2, 6_2, 7_3, 8_3).$$

All the operations T_{ik} transform plane pencils into quartic cones except when ik is one of the combinations 12, 34, 56, or 78. Thus we have

$$K_1(1)T_{12}K_5(2;\ 3_2, 4_2, 5_2, 6_2, 7_2, 8_2); \quad K_1(1)T_{13}K_4(3;\ 2_2, 4_1, 5_2, 7_1, 8_1).$$

Of the congruences of the second order, $K_1(1;\ 2_1)$ defines a particular $(2, 6)$; $K_1(1;\ 3_1)$ defines a particular $(2, 5)$, and $K_1(1;\ 7_1)$ a general $(2, 4)$.

Of the congruences of order 3, $K_1(1)$ defines a $(3, 7)$ having four cones of Σ, two pairs of planes, and a double quadric of the second kind. The cone $K_2(1;\ 6_1, 7_1, 8_1)$ defines a $(3, 7)$ congruence having a pair of planes for a double quadric of the second kind. The image cubic curve in the λ-plane has a double point at one node of $\Delta_4(\lambda)$ and passes simply through the other; it touches $\Delta_4(\lambda)$ in three other points.

19. The basis points may lie in three pairs of planes:

$$\pi_1 \equiv B_1B_2B_3B_4, \quad \pi_1' \equiv B_5B_6B_7B_8, \quad \pi_2 \equiv B_1B_2B_5B_6, \quad \pi_2' \equiv B_3B_4B_7B_8,$$
$$\pi_3 \equiv B_1B_3B_6B_8, \quad \pi_3' \equiv B_2B_4B_5B_7.$$

The locus of the vertices of cones in Σ now consists of the three lines $\pi_1\pi_1'$, $\pi_2\pi_2'$, $\pi_3\pi_3'$, and a space cubic, the latter being projected from B_1 by the cone $J_3(1; 4_1, 5_1, 7_2, 8_2)$. The involution I transforms a plane pencil into a cone of order 5 of the form $K_1(1)IK_5'(1; 2_1, 3_1, 4_2, 5_2, 6_1, 7_3, 8_2)$. The transformations T_{ik} can be most easily determined from the details of the $(3, 6)$ congruence defined by $K_1(1)$. The singular cones of both this congruence and its conjugate at B_1 are given above; the others are as follows:

$$K_4(2; 3_2, 4_1, 5_1, 6_2, 7_1, 8_2), \qquad K_4(3; 2_2, 4_1, 5_2, 6_2, 7_1, 8_1),$$
$$K_2'(2; 1_1, 7_1, 8_1), \qquad K_2'(3; 1_1, 7_1, 8_1),$$
$$K_3(4; 2_1, 3_1, 5_1, 6_2, 8_1), \qquad K_4(6; 2_2, 3_2, 4_2, 5_1, 7_1, 8_1),$$
$$K_3'(4; 1_2, 5_1, 6_1, 7_1, 8_1), \qquad K_2'(6; 1_1, 4_1, 7_1),$$
$$K_3(5; 2_1, 3_2, 4_1, 5_1, 8_1), \qquad K_3(8; 2_2, 3_1, 4_1, 5_1, 6_1),$$
$$K_3'(5; 1_2, 3_1, 4_1, 7_1, 8_1), \qquad K_3'(8; 1_2, 2_1, 4_1, 5_1, 7_1),$$
$$K_2(7; 2_1, 3_1, 5_1),$$
$$K_4'(7; 1_3, 2_1, 3_1, 4_1, 5_1, 6_1, 8_1).$$

The image of the congruence of cones is a trinodal quartic in the λ plane. Its four proper bitangents are images of the congruences cutting the lines l_{17}, l_{28}, l_{35}, l_{45} and their associated space cubics.

Of the congruences of order 2, $K_1(1; 7_1)$ defines the general $(2, 3)$ case. Its image in the λ-plane is a conic passing through all three nodes of $\Delta_4(\lambda)$ and touching it once. The cone $K_1(1; 8_1)$ defines a particular $(2, 4)$ congruence, and $K_1(1; 2_1)$ defines a particular $(2, 5)$. A $(2, 6)$ congruence cannot be contained in Σ in this case.

Among the congruences of order 3, that of lowest class is the $(3, 5)$ defined by $K_2(1; 5_1, 7_1, 8_1)$. It contains four of the basis planes simply and the remaining pair doubly. The section of the focal surface made by the latter consists of two conic sections, each counted twice. The image in the λ-plane is a cubic having a double-point at one of the double-points of $\Delta_4(\lambda)$, passing simply through each of the remaining nodes, and touching $\Delta_4(\lambda)$ in two other points.

As in the preceding cases, the non-hyperelliptic $K_6(1, 2_2, \cdots, 8_2)$ defines a $(4, 12)$ congruence with a singular cone of order 6 at each basis point, and containing 8 quadric cones of Σ.

20. The eight basis points may be arranged on four pairs of planes. In addition to the preceding pairs of planes π_i, π_i' we may write

$$\pi_4 \equiv B_1B_4B_5B_8, \qquad \pi_4' \equiv B_2B_3B_6B_7.$$

The locus of the vertices of cones now consists of the four lines $\pi_1\pi_1'$, etc., and

of their two transversals. The latter project from B_1 into $J_2(1;\ 7_2)$. The curve $\Delta_4(\lambda)$ consists of two conics passing through the images of the four lines $\pi_1\pi_1'$, etc. In the involution I,

$$K_1(1)IK_4'(1;\ 2_1, 3_1, 4_1, 5_1, 6_1, 7_3, 8_1),$$

that is, in the Cremona transformation defined by the involution I, the image of a plane passing through B_1 is in this case a quartic cone having its vertex at B_1, containing B_1B_7 as triple edge and passing simply through the remaining basis points.

The only congruences of the second order that can exist in this case are a particular $(2, 4)$ defined by $K_1(1;\ 2_1)$ and the general $(2, 2)$ defined by $K_1(1;\ 7_1)$. In the latter all the basis-points are vertices of plane pencils, and the conjugate congruence has the same configuration of singular planes. The image in the λ-plane consists of a conic belonging to the pencil defined by the factors of $\Delta_4(\lambda)$. The focal surface is the general Kummer surface.

Of congruences of the third order, $K_1(1)$ defines a $(3, 5)$ having a double quadric of the second kind; the elliptic $K_3(1;\ 2_1, 3_1, 4_1, 5_1, 6_1, 7_1)$ defines a $(3, 7)$ without a double quadric, and $K_3(1;\ 2_1, 3_1, 4_1, 5_1, 6_1, 8_1)$ defines a particular form of $(3, 9)$. When the cone is rational the corresponding congruence acquires a double quadric of the first kind.

As in the general case, the non-hyperelliptic $K_6(1;\ 2_2, \cdots, 8_2)$ defines a $(4, 12)$ congruence without any singular planes.

21. The basis points may be arranged on five pairs of planes, the four preceding, and

$$\pi_5 \equiv B_1, B_2, B_7, B_8, \qquad \pi_5' \equiv B_3, B_4, B_5, B_6.$$

The locus of the vertices of cones consists of the five lines $\pi_1\pi_1'$ etc., and of one other line, the latter projecting from B_1 into $J_1(1:7_1)$. The curve $\Delta_4(\lambda)$ consists of a conic and a pair of straight lines, each of the latter defining a pencil of quadric cones having common vertices. In the involution I, the image of a plane through B_1 is a cubic cone having B_1B_7 for double edge and passing simply through B_3, B_4, B_5, B_6. The symbol becomes

$$K_1(1)IK_3'(1:\ 3_1, 4_1, 5_1, 6_1, 7_2).$$

There are three particular congruences of the second order, a $(2, 4)$ defined by $K_1(1:2_1)$, a $(2, 3)$ by $K_1(1:3_1)$, and a $(2, 2)$ by $K_1(1:7)$.

Besides the rational $(3, 4)$ congruences defined by $K_1(1)$, the elliptic $(3, 6)$ defined by $K_3(1:2_1, 3_1, 4_1, 5_1, 6_1, 7_1)$ is of interest. For the first time the conjugate congruence of that defined by $K_6(1;\ 2_2, \cdots, 8_2)$ is not directly expressed by substitution of the values of the coördinates. The lines l_{11}, l_{18} are already three fold in the extraneous factors that divide out of $K_8'(1:\ 2_3, \cdots, 8_3)$, hence

are neutral. But by substituting the coördinates of these lines in K_6, both appear as double edges.

22. The basis points may be arranged on six pairs of planes, the five preceding and also

$$\pi_6 \equiv B_1 B_4 B_6 B_7, \qquad \pi_6' \equiv B_2 B_3 B_5 B_8.$$

The locus of the vertices of cones consists of the six lines $\pi_i \pi_i'$, four of which meet in a point. The cones having this point for vertex contain four common edges each containing two basis points, hence form a pencil belonging to Σ. The curve $\Delta_4(\lambda)$ consists of the sides of a quadrilateral. The involution I is defined by the transformation

$$K_1(1)\, I K_2'(1 : 3_1, 5_1, 7_1).$$

The plane $K_1(1)$ defines a $(3, 3)$ congruence having a double quadric of the first kind. It contains no cones, but has six pairs of planes. The conjugate congruence is of the same type. Congruences of order 3 and class 2 are impossible when the basis points are distinct. Particular $(2, 2)$ and $(2, 3)$ forms as well as $(3, k)$, $k = 3, \cdots, 6$, can be obtained. The involution reduces to ordinary quadratic inversion. As in the preceding cases $K_6(1 ; 2_2, \cdots, 8_2)$ defines a congruence of type $(4, 12)$, having a conjugate of the same kind.

When the points are arranged on six pairs of planes, they may be assumed to be $(\pm 1, \pm 1, \pm 1, 1)$. The equation of the bundle of quadrics is of the form

$$\lambda_1(x_1^2 - x_4^2) + \lambda_2(x_2^2 - x_4^2) + \lambda_3(x_3^2 - x_4^2) = 0,$$

and $\Delta_4(\lambda)$ becomes

$$\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3) = 0.$$

The equation of a line through $B_1 \equiv (1, 1, 1, 1)$ are $x_i = 1 + \rho \xi_i$, from which, if it belongs to Σ_1,

$$\lambda_1(\xi_1 - \xi_4) + \lambda_2(\xi_2 - \xi_4) + \lambda_3(\xi_3 - \xi_4) = 0,$$

$$\lambda_1(\xi_1^2 - \xi_4^2) + \lambda_2(\xi_2^2 - \xi_4^2) + \lambda_3(\xi_3^2 - \xi_4^2) = 0.$$

A congruence of the second order is defined by $K_1(1 ; 2_1)$, wherein $B_2 \equiv (1, -1, 1, 1)$. This cone requires relations of the form

$$A(\xi_1 - \xi_4) = B(\xi_2 + \xi_4) = C(\xi_3 - \xi_4)$$

among the ξ_i. The ξ-eliminant of these equations is the equation of the λ-curve, image of the congruence; it is the conic

$$(c\lambda_1 + A\lambda_3)^2 + \lambda_2(c^2\lambda_1 + A^2\lambda_3) = 0,$$

which passes through the vertices $(0, 1, 0)$, $(0, -1, 1)$, $(1, -1, 0)$ of the quadrilateral and touches the line $\lambda_2 = 0$ at $(A, 0, -c)$. The $(2, 3)$ congru-

ence contains one cone of Σ besides four singular cones whose vertices are at the basis points, and three pairs of planes besides the four singular pencils at four basis points. The conjugate is of the same form.

23. Instead of a finite number of basis points, the bundle Σ may have a basis curve. There are four general forms.

(1) When H_1, H_2, H_3 can be written in the form

$$u_1v_2 - u_2v_1 = 0,\ u_1w_2 - u_2w_1 = 0,\ v_1w_2 - v_1w_1 = 0,$$

wherein u_i, v_i, w_i are linear in the variables x_i, the bundle has a space cubic for basis curve.

(2) The equation

$$x_4(\lambda_1u_1 + \lambda_2u_2) + \lambda_3(x_1x_2 - x_3^2) = 0$$

defines a bundle of quadrics having the conics $x_4 = 0$, $x_1x_2 - x_3^2 = 0$ for basis curve and the two basis points $u_1 = 0$, $u_2 = 0$, $x_1x_2 - x_3^2 = 0$.

(3) The equation

$$\alpha x_1x_4 + \beta x_1x_3 + \gamma x_2x_3 + \delta x_2x_4 = 0,$$

in which α, β, γ, δ are linear homogeneous functions of λ_1, λ_2, λ_3, defines a bundle having the two basis lines $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, $x_4 = 0$.

(4) The bundle defined by

$$x_1u_4 - x_4u_1 = 0, \qquad x_2u_4 - x_4u_2 = 0, \qquad x_3u_4 - x_4u_3 = 0$$

has the line $x_4 = 0$, $u_4 = 0$ for basis curve. It has four basis points not lying on the line.

Any two quadrics contained in one of these forms (1), $\cdots$, (4) will intersect in the basis curve and a residual curve of order 1, 2, 2, or 3. In the first and third cases the residual intersection consists of one or of two generators. In the fourth case only will the residual intersection be a curve which is cut by every generator of one system in more than one point. On the focal surface of any congruence contained within the bundle the points of contact with a line of the congruence lie upon the complete intersection of two quadrics of the bundle. In the first two cases the lines are not proper bitangents; in the third the focal surface is a ruled surface contained within a linear congruence, while in the fourth the points of contact of lines of one congruence lie upon a bundle of space cubics of which they are bisecants. The operation of interchanging these points of contact is a birational transformation of order two. In case (3) it is simply an axial involution, hence case (4) will be the only one to be considered as defining a non-linear transformation.

24. The two systems of generators are rationally separable, hence the cubic complex is composite. One system may be written in the form

$$\frac{\lambda_1u_1 + \lambda_2u_2 + \lambda_3u_3}{u_4} = \frac{\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3}{x_4} = \sigma,$$

and the other becomes

$$\frac{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} = \frac{u_4}{x_4} = \sigma';$$

thus it is seen that Σ consists of a quadratic and of a special linear complex, the axis of the latter being the basis line $u_4 = 0$, $x_4 = 0$.

The four distinct basis-points are the vertices of the invariant tetrahedron of the collineation defined by

$$x_i = \rho u_i.$$

A congruence of order n is defined by a relation of the form $f_n(\lambda_1, \lambda_2, \lambda_3) = 0$.

As before, if ν is the class of f_n, the order of the focal surface will be 2ν; the line $u_4 = 0$, $x_4 = 0$ is an ν-fold line upon the surface, and each of the distinct basis points is also ν-fold upon it. Upon the focal surface there is moreover a cuspidal curve composed of as many space cubics as $f_n(\lambda)$ has points of inflexion.

One or more of the quadric cones contained in σ may break up into a pair of planes; one plane must contain the basis line and pass through one basis point, and the other contains the three remaining basis points, hence the maximum number of such degenerate cones is four.

The equation

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \sigma u_4$$

defines a pencil of planes through every tangent line to the curve $f_n(\lambda)$ in $u_4 = 0$. The equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \sigma x_4$$

defines a similar pencil through the tangents to the curve $f_n(\lambda)$ in $x_4 = 0$. From the equations $x_i = \rho u_i$ these two systems are projective and the congruence is defined as the locus of intersection of corresponding planes. The complex σ is thus a tetrahedral complex.

CORNELL UNIVERSITY,
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ON THE FUNDAMENTAL NUMBER OF THE ALGEBRAIC NUMBER-FIELD $k(\sqrt[p]{m})$

BY

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Introduction.

The object of the present paper is the determination of an integral basis and the fundamental number of the algebraic number-field $k(\sqrt[p]{m})$ generated by the real pth root of m, where m is a positive integer greater than unity which is not divisible by the pth power of an integer, and where p is any odd prime. The case $p = 3$ has already been discussed by DEDEKIND.* The conjugate values of $\sqrt[p]{m}$ being $\sqrt[p]{m}$, $\rho\sqrt[p]{m}$, $\cdots$, $\rho^{p-1}\sqrt[p]{m}$, where $\rho = e^{2\pi i/p}$, the number-fields $k(\rho\sqrt[p]{m})$, $\cdots$, $k(\rho^{p-1}\sqrt[p]{m})$ are all different from $k(\sqrt[p]{m})$.

In order to obtain all possible number-fields of this type we let m run through all positive integers which are not divisible by the pth power of a prime. But the fields generated in this way are not all distinct. For any positive integer m which is not divisible by the pth power of a prime may be expressed in one way only in the form

$$m = a_1 a_2^2 a_3^2 \cdots a_{p-1}^{p-1}$$

where $a_1 a_2 \cdots a_{p-1}$ is not divisible by the square of a prime. If we then set

$$\alpha_i = \sqrt[p]{a_1^{i_1} a_2^{i_2} a_3^{i_3} \cdots a_{p-1}^{i_{p-1}}}$$

where $i_s \equiv si \pmod{p}$ and $0 < i_s < p$ for $s = 1, 2, 3, \cdots, p-1$, it is evident that $\alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ are algebraic integers in $k(\alpha_1)$, and hence $k(\alpha_1), k(\alpha_2), \cdots, k(\alpha_{p-1})$ are identical, while $k(\alpha_1)$ is a primitive field.

1. *Rational basis.*

As a rational basis of $k(\alpha_1)$ we may take either

$$1, \alpha_1, \alpha_1^2, \cdots, \alpha_1^{p-1}$$

or

$$1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}.$$

**Über die Anzahl der Idealklassen in reinen kubischen Zahlkörpern*, Journal für die reine und angewandte Mathematik, vol. 121 (1899).

Denote the discriminants of these bases by D_1 and D_2, respectively. We have

$$D_1 = \begin{vmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{p-1} \\ 1 & \rho\alpha_1 & \cdots & \rho^{p-1}\alpha_1^{p-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & \rho^{p-1}\alpha_1 & \cdots & \rho^{(p-1)^2}\alpha_1^{p-1} \end{vmatrix}^2 = m^{p-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \rho & \cdots & \rho^{p-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & \rho^{p-1} & \cdots & \rho^{(p-1)^2} \end{vmatrix}^2$$

Hence *

$$D_1 = (-1)^{\frac{1}{2}(p-1)} p^p m^{p-1}.$$

In a similar way we obtain

$$D_2 = (-1)^{\frac{1}{2}(p-1)} p^p (a_1 a_2 \cdots a_{p-1})^{p-} .$$

If Δ be the fundamental number of $k(\alpha_1)$, we must have $D_2 = n^2\Delta$ where n is a rational integer. Hence

$$\Delta = (-1)^{\frac{1}{2}(p-1)} p \left[\frac{(pa_1 a_2 \cdots a_{p-1})^{\frac{1}{2}(p-1)}}{n} \right]^2 = (-1)^{\frac{1}{2}(p-1)} p d^2$$

where d is a rational integer, and this shows that Δ contains the factor p.

2. *Ideal Prime Factors of p and m.*

Let q be a prime factor of m and Q an ideal prime factor of q. Then since $\alpha_1^p = q^i r$, where r is prime to q and $0 < i < p$, it follows that α_1 is divisible by Q. Suppose that Q^s is the highest power of Q contained in q. Then α_1^p must be divisible by Q^{si} and $si \equiv 0 \pmod{p}$. Hence $s = p$ and $(q) = Q^p$, i. e., *every prime factor of m is equal to the pth power of a prime ideal of the first degree.*

Let us next consider the prime p. If p is a factor of m it comes under the case already considered. Suppose then that p is not contained in m. Since p is a factor of the fundamental number, it is divisible by the square of a prime ideal P. Now consider the integer $\mu = \alpha_1 - b$, where $b = a_1 a_2^2 \cdots a_{p-2}^{p-2}$. We have

$$(\mu + b)^p - b a_{p-1}^{p-1} = 0$$

or, if we set $d = b^{p-1} - a_{p-1}^{p-1}$,

$$\mu^p + pb\mu^{p-1} + \cdots + pb^{p-1}\mu + bd = 0.$$

Since $d \equiv 0 \pmod{p}$ it follows that μ^p is divisible by p and μ by P and hence d is divisible by P^3. Two cases arise according as d is divisible by p^2 or not.

I. *d not divisible by p^2.* In this case p must be divisible by P^3. Hence, if $p > 3$, d must be divisible by P^4 and therefore p divisible by P^4. Reasoning

* PASCAL, *Determinanten*, p. 139.

in this way we find that p must be divisible by P^p. Hence $(p) = P^p$, i. e., *if p is prime to m and $d = b^{p-1} - a_{p-1}^{p-1}$ not divisible by p^2, then p is equal to the p-th power of a prime ideal of the first degree.*

II. *d divisible by p^2.* Let $p^s (s \geqq 2)$ be the highest power of p contained in d and P^r the highest power of P contained in μ. The equation satisfied by μ may be written

$$\mu(\mu^{p-1} + p\beta) + bd = 0,$$

where β is prime to P. If r were greater than unity, μ^{p-1} would be divisible by a higher power of P than P^p, and since p cannot contain a higher power of P than P^p, it follows from the equation above that μ would be divisible by p. But if μ were divisible by p, its conjugates would be divisible by p, but this is impossible, since the coefficient of μ in the equation above contains only the first power of p. Hence $r = 1$. It is then easily seen that p must be divisible by P^{p-1} and by no higher power of p. *Hence if p is prime to m, and $b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2, we have $(p) = P^{p-1}Q$, where P and Q are different prime ideals of the first degree.*

3. *Integral basis.*

Any integer ω in $k(\alpha_1)$ ma be expressed in the form

$$\omega = \frac{x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{D_2},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. Let q be a prime factor of a_s and let $(q) = Q^p$. Then the highest power of Q contained in α_i is Q^{i_s}, where $i_s \equiv si \pmod{p}$ and $0 < i_s < r$. Hence x_0 must be divisible by Q and hence by q. Denote by $\alpha_{r_1}, \alpha_{r_2}, \cdots, \alpha_{r_{p-1}}$ the numbers $\alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ arranged according to increasing powers of Q. It then follows that x_{r_1} must be divisible by Q and hence by q. In the same way we find that $x_{r_2}, \cdots, x_{r_{p-1}}$ are divisible by q. It is then easily seen that ω may finally be written in the form

$$\omega = \frac{x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{p^p},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers.

If p is a factor of m we proceed as above and find that

$$\omega = y_0 + y_1\alpha_1 + \cdots + y_{p-1}\alpha_{p-1},$$

where $y_0, y_1, \cdots, y_{p-1}$ are rational integers.

If p is prime to m, two cases arise, according as $d = b^{p-1} - a_{p-1}^{p-1}$ is divisible by p^2 or not.

I. *d not divisible by p^2.* Introducing the algebraic integer $\mu = \alpha_1 - b$ mentioned above and making use of the fact that $\alpha_i = \alpha_1^i / c_i$, where c_i is a rational

integer prime to p, we obtain

$$c\omega = \frac{y_0 + y_1\mu + \cdots + y_{p-1}\mu^{p-1}}{p^p}.$$

In this case we have $(p) = P^p$ and, as is easily seen, μ is divisible by P but not by P^2. Reasoning in exactly the same way as above we find that $y_0, y_1, \cdots, y_{p-1}$ are all divisible by p. Hence we finally get

$$c\omega = z_0 + z_1\mu + \cdots + z_{p-1}\mu^{p-1},$$

where $z_0, z_1, \cdots, z_{p-1}$ are rational integers. But since all the prime factors of c are contained in m, it follows that ω may be written in the form

$$\omega = x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1},$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. We then have the following result:

If $b^{p-1} - a_{p-1}^{p-1}$ is not divisible by p^2, the p numbers $1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ form an integral basis of $k(\alpha_1)$ and $\Delta = D_2 = (-1)^{p-1\cdot 2}p^p(a_1a_2\cdots a_{p-1})^{p-1}$.

II. *d divisible by p^2.* In this case we know that $(p) = P^{p-1}Q$. We also know that μ^p is divisible by p, and hence μ is divisible by PQ and μ^{p-1} divisible by pQ^{p-2}. But

$$\mu^{p-1} = (\alpha_1 - b)^{p-1} = \alpha_1^{p-1} - (p-1)\alpha_1^{p-2}b \cdots + b^{p-1}$$

$$= [\alpha_1^{p-1} + \alpha_1^{p-2}b + \cdots \alpha_1 b^{p-2} + 1]$$

$$- \left[p\alpha_1^{p-2}b - \left\{\frac{(p-1)(p-2)}{2!} - 1\right\}\alpha_1^{p-3}b^2 + \cdots - b^{p-1} + 1\right]$$

and since $b^{p-1} \equiv 1 \pmod{p}$, it follows that

$$\gamma = \frac{\alpha_1^{p-1} + \alpha_1^{p-2}b + \cdots + \alpha_1 b^{p-2} + 1}{p}$$

is an algebraic integer. We shall now prove that the p numbers

$$\gamma, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$$

form an integral basis of $k(\alpha_1)$. It is evident that these numbers form a rational basis. Denoting the discriminant of this basis by D_3 we get the following value

$$D_3 = (-1)^{p-1/2}p^{p-2}(a_1a_2\cdots a_{p-1})^{p-1}.$$

Now any algebraic integer ω may be written in the form

$$\omega = \frac{x_0\gamma + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{D_3}$$

where $x_0, x_1, \cdots, x_{p-1}$ are rational integers. It is easily seen that x_0 must be

divisible by D_3. For, denoting by $\omega, \omega', \cdots, \omega^{(p-1)}$ the conjugate values of ω, we have

$$\omega + \omega' + \cdots + \omega^{(p-1)} = \frac{x_0}{D_3}.$$

Hence x_0 is divisible by D_3. Let us then consider the algebraic integer

$$\omega_1 = \frac{x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}}{D_3}.$$

If q be a prime factor of m we infer in the same way as above that $x_1, \cdots, x_{p-1}$ are divisible by q, and hence that ω_1 may be written in the form

$$\omega_1 = \frac{y_1\alpha_1 + \cdots + y_{p-1}\alpha_{p-1}}{p^{p-2}}.$$

Replacing α_1 by $\mu + b$ we get

$$p^{p-2}c\omega_1 = z_0 + z_1\mu + \cdots + z_{p-1}\mu^{p-1}$$

where $z_0, z_1, \cdots, z_{p-1}$ are rational integers and c is prime to p. By a simple argument it can then be shown that $z_0, z_1, \cdots, z_{p-1}$ and hence also $y_1, y_2, \cdots, y_{p-1}$ must be divisible by p and that ω may finally be written in the form

$$\omega = x_0\gamma + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}.$$

Hence we have the following result: *If* $b^{p-1} - a_{p-1}^{p-1}$ *is divisible by* p^2, *the* p *numbers* $\gamma, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ *form an integral basis of* $k(\alpha_1)$ *and*

$$\Delta = D_3 = (-1)^{p-1/2}p^{p-2}(a_1a_1\cdots a_{p-1})^{p-1}.$$

PURDUE UNIVERSITY.

VOLTERRA'S INTEGRAL EQUATION OF THE SECOND KIND, WITH DISCONTINUOUS KERNEL*

BY

GRIFFITH C. EVANS

The integral equation of the second kind, of VOLTERRA, is written

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi. \tag{1}$$

In this equation the function $K(x, \xi)$ is called the kernel; the desired function is $u(x)$. When the functions $K(x, \xi)$ and $\phi(x)$ are continuous there is no difficulty in finding a continuous $u(x)$ that shall satisfy equation (1). This general case of the equation (1), under the conditions solely that $\phi(x)$ be continuous when $a \leqq x \leqq b$ and that $K(x, \xi)$ be continuous in the triangular region $a \leqq \xi \leqq x \leqq b$, was first investigated by VOLTERRA,† who showed that there is one and only one continuous solution in the interval $a \leqq x \leqq b$. His method applies without essential change to equations whose kernels are finite in the region $a \leqq \xi \leqq x \leqq b$ and have discontinuities, provided the discontinuities are regularly‡ distributed.

Let us consider, however, a certain equation to which we are led by a hydrostatical problem. Suppose we are given a tube lying in a vertical plane along a curve of arbitrary shape, $s = u(x)$, where s is the distance along the curve and x the altitude. Let us fill this tube with a liquid of variable linear density ν, and then regulate its height x in the tube by allowing various amounts to flow through the bottom. Let us then regard ν as an analytic function of the depth in the liquid, i. e. $\nu = \nu(x - \xi)$. The average linear density is given by the formula

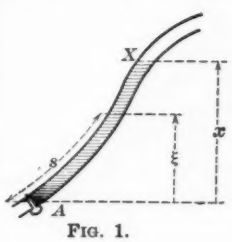

FIG. 1.

$$h(x) = \frac{\int_A^X \nu(x - \xi) ds}{\int_A^X ds} = \frac{\int_0^x \nu(x - \xi) u'(\xi) d\xi}{\int_0^x u'(\xi) d\xi},$$

* Presented to the Society, September 13, 1909.

† V. VOLTERRA, Atti dell' Accademia dei Lincei, ser. 5, vol. 5 (1896), pp. 177–184.

‡ M. BÔCHER: *An Introduction to the Study of Integral Equations;* Cambridge Tracts in Mathematics and Mathematical Physics, No. 10, 1909.

so that, when the curve is given, the average density is determined explicitly. Let us propose to ourselves the inverse problem: given the average linear density, to find the shape of the curve.

In the first place, if $u'(\xi)$ is continuous $h(x)$ is continuous, and if we let

$$\begin{aligned} \nu(x-\xi) &= \alpha + \alpha_1(x-\xi) + \alpha_2(x-\xi)^2 + \cdots \\ &= \alpha + G(x-\xi), \end{aligned}$$

where $G(0)=0$, then $h(x)=\alpha+g(x)$, where $g(x)$ is some function that vanishes at least to the first order when $x=0$. We shall look only for solutions $u(x)$ that have continuous derivatives. Hence we may write the equation to determine $u'(x)$ as

$$g(x)\int_0^x u'(\xi)d\xi = \int_0^x G(x-\xi)u'(\xi)d\xi,$$

or

$$0 = \int_0^x [G(x-\xi)-g(x)]v(\xi)d\xi,$$

where $v(\xi)=u'(\xi)$.

This is an equation of which the kernel vanishes when $x=\xi=0$. If we differentiate once and divide by $g(x)$, we get the equation

$$v(x) = \int_0^x \frac{G(x-\xi)-g'(x)}{g(x)} v(\xi)d\xi,$$

which is a homogeneous equation of the form (1), with *discontinuous* kernel.*

The corresponding extension in the general equation (1) of the second kind is found by differentiation of the general integral equation of the first kind

$$\phi(x)-\phi(a) = \int_a^x K(x,\xi)u(\xi)d\xi. \tag{2}$$

It was originally an equation of this kind,

$$\psi(a) = \int_{x=0}^{x=a} \frac{ds}{(a-x)^n}, \qquad 0<n<1,$$

by which ABEL in 1823† introduced the subject of integral equations; an equation of the first kind with discontinuous kernel. The general equation (2) where the kernel is continuous is itself not easy to treat. Yet in his first paper VOLTERRA shows that if $\phi(y)$ and $\phi'(y)$ remain continuous when $a \leqq y \leqq b$, and if $K(x,y)$ and $\partial K/\partial x = K_1(x,y)$ remain continuous in the triangular

* This equation will be treated in a later paper.

† VOLTERRA: Annali di Matematica, ser. 2, vol. 25 (1897), p. 140.

region $a \leqq y \leqq x \leqq b$, while $K(x, x) \neq 0$, $a \leqq x \leqq b$, then there is one and only one solution of (2) continuous in the interval $a \leqq x \leqq b$.*

There are two manifest ways of enlarging the kernel of (2) beyond VOLTERRA's first restrictions. One way is to generalize the equation of ABEL to the form

$$\phi(x) - \phi(a) = \int_a^x \frac{G(x, \xi)}{(x - \xi)^\lambda} u(\xi) d\xi, \qquad \lambda < 1,$$

where $G(x, \xi)$ and $\partial G/\partial x$ are continuous in the triangular region and $G(x, x) \neq 0$; this generalization VOLTERRA carries out in the second of his papers† appearing in the Atti di Torino, and shows that there is one and only one continuous solution. A second way is to allow the kernel of (2) to vanish when $x = \xi = a$. This generalization, with the restriction that $K(x, \xi)$ shall not vanish to higher order when ξ and x are equal and both approach a, than when ξ and x approach a independently, is considered by VOLTERRA in his third and fourth Turin papers.‡ And here it turns out that there is in general more than one solution continuous in the neighborhood of a.

The same problem is considered by E. HOLMGREN§ in 1900, and the special case is treated in which the kernel is in the form

$$K(x, y) = a_0 y + a_1 x + K'(x, y)$$

and $a_0 + a_1$ is not necessarily unequal to zero. The present state of the question is best set forth in a recent paper of LALESCO.‖ He shows among other things that the equation

$$\int_0^x f(x, s) u(s) ds = F(x), \tag{3}$$

in which

$$f(x, s) = A_n x^n + A_{n-1} x^{n-1} s + \cdots + A_0 s^n + \psi(x, s),$$

— here $\psi(x, s)$ contains only terms whose total degree in x and s is greater than n, and $A_0 + A_1 + \cdots + A_n \neq 0$, — has a solution $u(x)$, finite at $x = 0$ and depending linearly on k parameters, provided that $F(x)$ vanishes to a sufficiently high order and that k distinct solutions of the equation

*Atti della Reale Accademia delle Scienze di Torino, vol. 31, Jan., 1896, pp. 311–323.

†Atti della Reale Accademia delle Scienze di Torino, vol. 31, Jan., 1896, pp. 400–408.

‡Atti della Reale Accademia delle Scienze di Torino, vol. 31, March, 1896, pp. 557–567 and April, 1896, pp. 693–708.

§E. HOLMGREN: Sur une théorème de M. Volterra sur l'inversion des intégrales définies; Atti della Reale Acc. delle Scienze di Torino, vol. 35 (1900), p. 570.

‖T. LALESCO: Sur l'équation de Volterra; Journal de Mathématiques, ser. 6, vol. 4 (1908) p. 125.

$$\frac{A_0}{r-1}+\frac{A_1}{r-2}+\cdots+\frac{A_n}{r-n-1}=0\,*$$

have their real parts negative. LALESCO considers also the special case in which

$$f(x,s)=A_1x+A_0s+\alpha x^2+\beta xy+\gamma y^2+\psi_3(x,s),$$

where $A_1+A_0=0$ and $\alpha+\beta+\gamma\neq 0$, and shows, as was previously announced by E. HOLMGREN (see page 395, note), that if

$$\frac{A_0}{\alpha+\beta+\gamma}>0$$

there will be a single real solution, while if

$$\frac{A_0}{\alpha+\beta+\gamma}<0$$

there will be an infinity of solutions depending on a single parameter.

LALESCO's method of procedure, to which the method of our later paper is analogous, is to differentiate the equation (3) in the general case $n+1$ times, and then to base on the resulting Fuchsian equations an approximation method by which the given equation may be solved. It is essential in order to reach a Fuchsian equation that $A_0+A_1+\cdots+A_n\neq 0$.

The equation to which (3) is reduced by differentiating more than once is a special case of a type of equations proposed by BURGATTI,† and investigated in a short note by FUBINI,‡ i. e., equations of the form

$$(4)\quad \int_0^x\left[f_0(x,s)u(s)+f_1(x,s)\frac{du}{ds}+\cdots+f_n(x,s)\frac{d^nu}{ds^n}\right]ds=F(x).$$

The connection between equations of the first kind with kernel vanishing at the initial point of the interval, and equations of the second kind with kernels discontinuous at the initial point of the interval is seen, as we have said, by differentiating once. Equation (3) thus becomes

$$f(x,x)u(x)+\int_0^x\frac{\partial f(x,s)}{\partial x}u(s)\,ds=F'(x)$$

or

$$u(x)=\frac{F'(x)}{f(x,x)}-\int_0^x\frac{1}{f(x,x)}\frac{\partial f(x,s)}{\partial x}u(s)\,ds,$$

* This equation was originally given by VOLTERRA, and the fact shown that there is but one finite solution if the real parts of all the roots are positive, while otherwise the solution is not determined uniquely.

† BURGATTI: Rendiconti dell' Accademia dei Lincei, 2. semestre 1903, p. 443 and p. 596.

‡ FUBINI: Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli, ser. 3, vol. 10 (1904), pp. 61–64.

which on account of the restriction $A_0 + A_1 + \cdots + A_a \neq 0$ is an equation the kernel of which becomes infinite at $x = \xi = 0$ to not higher than the first order. The special equation on page 396 corresponds to a case where the kernel of an equation of the second kind becomes infinite at $x = \xi = 0$ to the second order.

In the present paper we consider equations whose kernels are absolutely integrable, and equations related to them by change of dependent variable. Later we shall treat equations whose kernels, though not of so general a type, are not restricted in their order of becoming infinite, and endeavor to define as to continuity the character of functions that can possibly satisfy the equations under investigation.

Throughout the course of the work it will be convenient to be able to refer to the following two conditions, (A) and (B).

Condition (A). A real function of the two variables x, ξ is to be continuous in the triangle $T: a \leqq \xi \leqq x \leqq b,\ b > a \geqq 0$ except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number, $x = \beta_1, x = \beta_2, \cdots, x = \beta_r$. On the other portions of the system of curves there are to be only a finite number of vertical tangents. Hence by any line $x = x_0, x_0 \neq \beta_1, \cdots, \beta_r$, the system of curves will be cut in only a finite number of points.

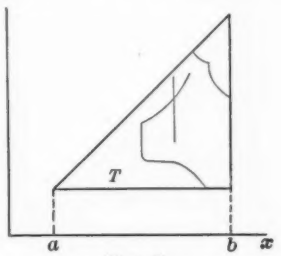

FIG. 2.

Condition (B). In the region $t: a \leqq x \leqq b$, a real function of the single variable x is to be continuous except at a finite number of points, $\gamma_1, \cdots, \gamma_s$, and is to remain finite.

§ 1. *A General Theorem.*

The existence of the unique continuous solution of (1) when $\phi(x)$ and the kernel are continuous is established by the development of $u(x)$ into a series by substitution of the equation into itself.* If we assume for the moment that the equation (1) has a continuous solution, $u(x)$, then the following equations will hold:

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi,$$

$$u(x) = \phi(x) + \int_a^x K(x, \xi)\phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') u(\xi') d\xi' d\xi,$$

* Method of Successive Substitutions, LIOUVILLE: Journal de Mathématiques, vol. 2 (1837), pp. 16–36 (applied to a particular equation).

etc. As n becomes infinite, the general expression becomes an infinite series. This series is shown to be uniformly convergent, and a solution of the given integral equation (1). The same method applies when $K(x, \xi)$ is not so much restricted.*

Let us define besides the linear region t and the two dimensional region T, a linear region t_δ formed from t by removing the small portions $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$; and a two dimensional region T_δ formed from T by removing the small strips $\alpha_i - \delta < x < \alpha_i + \delta$ $(i = 1, 2, \cdots, l)$, where the δ is an arbitrarily small magnitude, and the α's are finite in number.

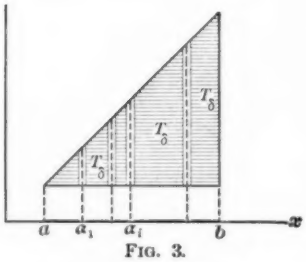

FIG. 3.

THEOREM. *Consider the integral equation* (1). *Let* $K(x, \xi)$ *satisfy Condition* (A), *and* $\phi(x)$ *Condition* (B). *Furthermore let us assume the following conditions:*

(a) $\int_a^x |K(x, \xi)|\, d\xi$ *converges for all values of* x *in the interval* $a \leqq x \leqq b$ *except for a finite number of such values,*† $x = \lambda_1, \lambda_2, \cdots, \lambda_i$, *and the function thus defined remains finite, less than* M;

(b) *Let* ϵ *and* δ *be two arbitrary positive quantities. Then there is a positive quantity* η *and a region* T_δ *such that*

$$\int_y^{y+\eta} |K(x, \xi)|\, d\xi < \epsilon, \qquad (x, y) \text{ and } (x, y + \eta) \text{ in } T_\delta.$$

The points $\alpha_1, \cdots, \alpha_i$ *that appear in the definition of* T_δ *are independent of* ϵ *and* δ *and include the points* β *of* (A) *and* λ *of* (a);

(c) t *can be divided into* k *parts, bounded by the points* $a = a_0, a_1, \cdots, b = a_k$, *such that*

$$\int_{a_i}^x |K(x, \xi)|\, d\xi \leqq H < 1 \qquad a_i \leqq x \leqq a_{i+1};\ x \neq a_1, \cdots, a_l.$$

Under these conditions there is one and only one solution $u(x)$ *of the integral equation* (1), *continuous in* t *except for a finite number of points, and these points will be among the points* $\gamma_1, \cdots, \gamma_s$, *of* (B), *and* $\alpha_1, \cdots, \alpha_l$, *of* (b).

In the proof of this theorem (a) and (c) are used in showing the convergence of the expansion of the solution, and (b) in developing what continuity exists.

* The principal theorem of this part is a generalization of theorems submitted by Dr. W. A. HURWITZ, Dr. C. N. MOORE and myself as answers to a problem given out by Professor MAXIME BÔCHER in his course on integral equations, Harvard University, 1907–08.

† An example of a point λ not a point β is given by the point $x = 1$, where $K(x, \xi) = 1$ in the horizontally shaded region of the adjoined figure, and $K(x, \xi) = 1/(x - 1)$ in the vertically shaded region.

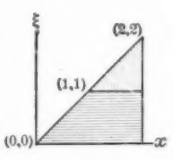

FIG. 4.

§ 2. *Convergence of the Expansion.*

We shall show that the series

$$\bar{u}(x) = \int_a^x K(x, \xi)\phi(\xi)d\xi + \int_a^x K(x, \xi)\int_a^\xi K(\xi, \xi')\phi(\xi')d\xi' d\xi + \cdots$$

converges absolutely and uniformly in the region t_δ and represents in t a function finite and continuous except for a finite number of points.

Every term of this series is continuous in the region t_δ. For if $r(x)$ is any function continuous in t except for a finite number of points and finite, in absolute value less than N, the integral

$$F(x) = \int_a^x K(x, \xi)r(\xi)d\xi$$

is continuous in the region t_δ and finite in t.

To prove this take x_0, any point in t_δ, and $x_0 + \Delta x$, a point near it and in t_δ, Fig. 5, and consider

$$\left|\int_a^{x_0+\Delta x} K(x_0+\Delta x, \xi)r(\xi)d\xi - \int_a^{x_0} K(x_0, \xi)r(\xi)d\xi\right| = |\Delta F|$$

$$\leqq \left|\int_{x_0}^{x_0+\Delta x} K(x_0+\Delta x, \xi)r(\xi)d\xi\right| + \left|\int_a^{x_0} \{K(x_0+\Delta x, \xi) - K(x_0, \xi)\}r(\xi)d\xi\right|$$

$$\leqq N\int_{x_0}^{x_0+\Delta x} |K(x_0+\Delta x, \xi)|\,|d\xi| + N\int_a^{x_0} |K(x_0+\Delta x, \xi) - K(x_0, \xi)|\,|d\xi|.$$

Let us assign the value ρ, arbitrarily small, in advance. On each side of every one of the p points $\xi_1, \xi_2, \cdots, \xi_p$ in which the line $x = x_0$ is cut by the system of curves of discontinuity of $K(x, \xi)$ lay off a distance $\frac{1}{2}\eta$ such that

$$\int_{\xi_i-\frac{1}{2}\eta}^{\xi_i+\frac{1}{2}\eta} |K(x, \xi)|\,|d\xi| < \epsilon$$

in accordance with the condition (b) of the theorem. Take

$$\epsilon = \left(\frac{\rho}{p+1}\right)\left(\frac{1}{2N}\right).$$

Take $\Delta x < \eta$, and small enough so that none of the curves of the system are cut by any of the lines $\xi = \xi_i \pm \eta/2$ in the interval $x_0 \leqq x \leqq x_0 + \Delta x$. We shall then have

$$|\Delta F| \leqq \frac{\rho}{2} + N\left[\Sigma\int_{A_i}^{B_i} |K(x_0+\Delta x, \xi) - K(x_0\xi)|\,|d\xi|\right],$$

where the integration is now extended over the sections of the line $x = x_0$

remaining after the p pieces have been removed. And since $K(x, \xi)$ is continuous in these closed regions (the shaded regions in the diagram, Fig. 5), Δx may be chosen small enough so that

$$|K(x_0 + \Delta x, \xi) - K(x, \xi)| \leqq \frac{\rho}{(x_0 - a)} \frac{1}{2N}.$$

Accordingly we have, finally,

$$|\Delta F| \leqq \frac{\rho}{2} + \frac{\rho}{2(x-a)} \Sigma \{B_i - A_i\} \leqq \frac{\rho}{2} + \frac{\rho}{2(x-a)} (x-a) \leqq \rho;$$

and F is continuous in t_δ, i. e. continuous in t except for the points $\alpha_1, \cdots, \alpha_l$, since δ is as small as we please to take it. Moreover, F is finite in t since $|F| \leqq NM$.

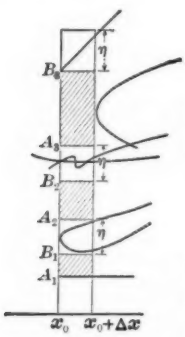

FIG. 5.

The same proof applies to the function

$$R(x) = \int_a^R K(x, \xi) r(\xi) d\xi, \qquad a \leqq R \leqq x,$$

showing that it is continuous except for the points α and finite, in t.

The function

$$\int_a^x K(x, \xi) r(\xi) d\xi$$

itself satisfies the conditions laid on $r(x)$. For it is continuous except for the points α and finite. Consequently, as is seen by mathematical induction, every term, and every integral appearing in every term, of the series is an $r(x)$, with points of discontinuity included in the points $\alpha_1, \cdots, \alpha_l$.

We now show the absolute and uniform convergence of the series

$$(3) \quad \bar{u}(x) = \int_a^x K(x, \xi) \phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \phi(\xi') d\xi' d\xi + \cdots$$

in the region t_δ. If we write it as

$$\sum_{n=0}^{\infty} u_{n+1}(x)$$

its general term is

$$u_{n+1}(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} K(\xi^{(n-1)}, \xi^{(n)}) \phi(\xi^{(n)}) d\xi^{(n)} \cdots d\xi' d\xi.$$

We have immediately

$$(4) \quad |u_{n+1}(x)| \leqq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \cdots d\xi' d\xi,$$

and in that part of the region t_δ for which x lies in the first of the intervals mentioned in (c), i. e., $a \leqq x \leqq a_1$, it follows that

$$|u_{n+1}(x)| \leqq NH^{n+1},$$

independently of δ. We may write this, for convenience, as

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{j-1}H^{n+2-j}, \qquad j=1. \tag{5}$$

We can establish corresponding inequalities of the type (5) for any interval a_{i-1} to a_i, by a process of mathematical induction. For if we assume the formula

$$\begin{aligned} |u_{n+1}(x)| &\leqq N\int_a^x |K(x,\xi)|\int_a^\xi |K(\xi,\xi')|\int_a^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi' d\xi \\ &\leqq N\{(n+1)M\}^{j-1}H^{n+2-j} \end{aligned} \tag{6}$$

to hold independently of the value of δ for the first $i-1$ intervals of t, i. e., when $j = 1, 2, \cdots, i-1$, respectively, we can show that it holds for the ith interval, $j = i$, as well. It is obvious, since $M \geqq H$ and $|u_{n+1}(x)| \leqq NM^{n+1}$, that the formula holds in the ith interval when $n \leqq i-2$. Also, since $M \geqq H$,

$$N\{(n+1)M\}^{j_1-1}H^{n+2-j_1} \leqq N\{n+1)M\}^{j_2-1}H^{n+2-j_2}$$

if $j_2 > j_1$; hence the formula

$$\begin{aligned} |u_{n+1}(x)| &\leqq N\int_a^x |K(x,\xi)|\int_a^\xi |K(\xi,\xi')|\int^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|d\xi^{(n)}\cdots d\xi' d\xi \\ &\leqq N\{(n+1)M\}^{i-2}H^{n+3-i} \qquad (j=i-1), \end{aligned} \tag{7}$$

holds if x is in any of the first $i-1$ intervals.

Now, by rearrangement of the integral in (4) we have

$$\begin{aligned} |u_{n+1}(x)| \leqq\ & \int_a^{a_{i-1}} |K(x,\xi)|\left\{N\int_a^\xi |K(\xi,\xi')|\int_a^{\xi'}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi'\right\}d\xi \\ &+\int_{a_{i-1}}^x |K(x,\xi)|\left[\int_a^{a_{i-1}} |K(\xi,\xi')|\left\{N\int_a^{\xi'} |K(\xi',\xi'')|\int_a^{\xi''}\right.\right. \\ &\left.\left.\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi''\right\}d\xi'\right]d\xi+\int_{a_{i-1}}^x |K(x,\xi)|\int_{a_{i-1}}^\xi |K(\xi,\xi')| \\ &\times\left[\int_a^{a_{i-1}} |K(\xi',\xi'')|\left\{N\int_a^{\xi''}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|d\xi^{(n)}\cdots d\xi'''\right\}d\xi''\right]d\xi' d\xi \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+N\int_{a_{i-1}}^x |K(x,\xi)|\int_{a_{i-1}}^\xi |K(\xi,\xi')|\int_{a_{i-1}}^{\xi'}\cdots\left[\int_{a_{i-1}}^{\xi^{(n+1-i)}} |K(\xi^{(n+1-i)},\xi^{(n+2-i)})|\right. \\ &\left.\times\int_a^{\xi^{(n+2-1)}}\cdots\int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)},\xi^{(n)})|\,d\xi^{(n)}\cdots d\xi^{(n+2-i)}\right]d\xi^{(n+1-i)}\cdots d\xi. \end{aligned}$$

These $n+3-i$ terms we have already a means for treating. For, as we have shown on page 401, we can apply to the integrals within the $\{\ \}$ the formula (7). For the single remaining integral inside the square bracket, in the first $n+2-i$ terms, we may write

$$\int_a^{a_{i-1}} |K(\xi^{(r)}, \xi^{(r+1)})|\, d\xi^{(r+1)} \leqq M.$$

The terms outside both brackets we may write as less than or equal to a power of H. The last term we have directly $\leqq NH^{n+2-i}M^{i-1}$. We may write, then, for the ith interval of t_δ

$$|u_{n+1}(x)| \leqq$$

$$\begin{aligned}
&MN\{(n-1+1)M\}^{i-2}H^{n-1+3-i} && (\text{or } Nn^{i-2}M^{i-1}H^{n+2-i})\\
&+HMN\{(n-2+1)M\}^{i-2}H^{n-2+3-i} && (\text{or } N(n-1)^{i-2}M^{i-1}H^{n+2-i})\\
&+H^2MN\{(n-3+1)M\}^{i-2}H^{n-3+3-i} && (\text{or } N(n-2)^{i-2}M^{i-1}H^{n+2-i})\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+H^{n+1-i}MN\{(i-1)M\}^{i-2}H && (\text{or } N(i-1)^{i-2}M^{i-1}H^{n+2-i})\\
&+NM^{i-1}H^{n+2-i}.
\end{aligned}$$

Hence we have

$$|u_{n+1}(x)| \leqq NM^{i-1}H^{n+2-i}[n^{i-2}+(n-1)^{i-2}+(n-2)^{i-2}+\cdots+(i-1)^{i-2}+1],$$

where there are in the bracket $n+3-i$ terms. Hence, since $i \geqq 2$,

$$|u_{n+1}(x)| \leqq NM^{i-1}H^{n+2-1}(n+1)n^{i-2} \leqq NM^{i-1}H^{n+2-i}(n+1)(n+1)^{i-2},$$

and finally,

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{i-1}H^{n+2-1},$$

the desired result, which holds now not only for the ith interval of t_δ, but also for any point x in the first i intervals. Hence we have for any point x of the k intervals of t_δ,

$$|u_{n+1}(x)| \leqq N\{(n+1)M\}^{k-1}H^{n+2-k},$$

a value independent of the magnitude of δ. The series of constants, composed of terms of the type of the right hand side, is convergent by the simplest test. Suppose its value is Σ; then

$$|u(x)| \leqq \Sigma,$$

where Σ is independent of δ. Since the series is a uniformly convergent series of continuous functions in t_δ, it represents in t_δ a continuous function, regardless of the value of δ, $(\delta > 0)$.

§ 3. *Existence of a Solution.*

If we define

$$U(x) = \phi(x) + \bar{u}(x) \tag{8}$$

we know that $U(x)$ is a function continuous in t except for a finite number of points, which are among the points $\gamma_1, \cdots, \gamma_\nu, \alpha_1, \cdots, \alpha_l$; and finite $\leqq \Sigma + N$. It is now easy to show that $U(x)$ is a solution of the integral equation (1).

If we write for this purpose

$$U(x) = S_n(x) + R_n(x),$$

where $S_n(x)$ is the sum of the first n terms, the three functions $U(x)$, $S_n(x)$, $R_n(x)$ are of the type $r(x)$, § 2, and

$$\int_a^x K(x, \xi)U(\xi)d\xi = \int_a^x K(x, \xi)S_n(\xi)d\xi + \int_a^x K(x, \xi)R_n(\xi)d\xi$$

or

$$\int_a^x K(x, \xi)U(\xi)d\xi - \int_a^x K(x, \xi)S_n(\xi)d\xi = \int_a^x K(x, \xi)R_n(\xi)d\xi.$$

Hence

$$\lim_{n=\infty}\left(\int_a^x K(x, \xi)U(\xi)d\xi - \int_a^x K(x, \xi)S_n(\xi)d\xi\right) = 0$$

for

$$\left|\int_a^x K(x, \xi)R_n(\xi)d\xi\right| \leqq \int_a^x |K(x, \xi)||R_n(\xi)|d\xi;$$

and since $|R_n(x)|$ can be made less than an arbitrarily assigned ϵ by a suitable choice of n, we have

$$\left|\int_a^x K(x, \xi)R_n(\xi)d\xi\right| \leqq \epsilon\int_a^x |K(x, \xi)|d\xi,$$

which in t_δ is $\leqq \epsilon M$ and can therefore be made as small as we please by taking n large enough.

Hence

$$\int_a^x K(x, \xi)U(\xi)d\xi = \lim_{n=\infty}\int_a^x K(x, \xi)S_n(\xi)d\xi$$

and we can integrate the series term by term. We see in this way that $U(x)$ and

$$\phi(x) + \int_a^x K(x, \xi)U(\xi)d\xi$$

are identical, and that consequently $U(x)$ is a solution of the integral equation (1).

§ 4. *Uniqueness of the Solution.*

If $u(x)$ is any solution whatever, finite, and continuous in t except for a finite number of points,

$$v(x) = u(x) - U(x)$$

will be a function of the same kind, and since

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi$$

and

$$U(x) = \phi(x) + \int_a^x K(x, \xi) U(\xi) a\xi$$

$v(x)$ will be a solution of the homogeneous integral equation

$$v(x) = \int_a^x K(x, \xi) v(\xi) d\xi.$$

Consequently

$$v(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} K(\xi^{(n-1)} \xi^{(n)}) d\xi^{(n)} \ldots d\xi, \qquad n = 1, 2, \cdots,$$

and

$$|v(x)| \leqq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \cdots \int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \ldots d\xi.$$

But by the analysis of § 2, this last inequality leads to

$$|v(x)| \leqq N \{(n+1) M\}^{k-1} H^{n+2-k}, \qquad x \text{ in } t_\delta,$$

and since $H < 1$ and n is arbitrary, the right hand side can be made as small as we please. Hence for any point of t_δ

$$v(x) = 0$$

and $U(x)$ is the only solution of the given type. Thus the theorem of § 1 is proved.

A few examples will show its scope. A kernel $K(x, \xi)$ that satisfies the given conditions is $1/2(x - a)$. Here

$$\int_a^x |K(x, \xi)| d\xi = \tfrac{1}{2},$$

and we know that there is one and only one finite solution of

$$u(x) = \phi(x) + \int_a^x \frac{u(\xi)}{2(x - a)} d\xi$$

through any finite range, a to b, whatever. A more complicated example of the same kind is

$$K(x,\xi)=\begin{cases}\dfrac{1}{5(x-a)}, & \cdots\cdots, \quad a\leqq x\leqq x_0;\\ \dfrac{1}{5(x_0-a)}, & \cdots, \quad a\leqq\xi\leqq x_0\\ \dfrac{1}{5(x_0-a)}+\dfrac{1}{2(x-x_0)}, & x_0<\xi\leqq x\end{cases}\Bigg\}\ x_0<x\leqq b.$$

Other examples, of theoretical importance, will appear later. An example where the theorem does not hold we get by putting $K(x,\xi)=2/(x-a)$. This fails to satisfy (c), for

$$\int_a^x |K(x,\xi)|\,d\xi = \infty$$

and we can choose no first interval small enough so that

$$\int_a^x |K(x,\xi)|\,d\xi \leqq H<1$$

within it. As a matter of fact, in this case there is more than one finite solution. A second case, where (c) is satisfied but where (a) fails to be, is obtained by choosing

$$K(x,\xi)=\begin{cases}\dfrac{1}{5(x-a)}, & a\leqq x\leqq x_0;\\ \dfrac{1}{5(\xi-a)}, & x_0<x\leqq b.\end{cases}$$

§ 5. *Related Theorems.*

There are other theorems more or less closely related to the one that we have developed, as extensions or as special cases. Practically the same proof holds when we replace (A), (B), (a), (b), (c), by:

(A') $K(x,\xi)$ is continuous in general* in T except in the neighborhood of a system of curves of the kind specified in Condition (A).

(B') $\phi(x)$ is continuous in general in t except in the neighborhood of a finite number of points, and is finite.

* By "$K(x,\xi)$ is continuous in general except in the neighborhood of a system of curves of the kind specified in Condition (A)," we mean that there exists a finite number of curves of the kind specified in Condition (A) such that if every one is enclosed by a strip of width 2ρ, i. e., ρ on each side, where ρ is arbitrarily small, and a region T_ρ formed from T by taking away these strips, then $K(x,\xi)$ will be continuous throughout T_ρ except for a finite number of curves of the kind specified in Condition (A). An analogous meaning is attached to "$\phi(x)$ continuous in general in t except in the neighborhood of a finite number of points." Thus $1/\sin 1/x$ is "continuous in general except in the neighborhood" of the point $x=0$, Fig. 6.

(a') The integral

$$\int_a^x |K(x, \xi)|\, d\xi$$

converges in t except in the neighborhood of a finite number of points $\lambda_1, \cdots, \lambda_l$, and remains finite.

(b') There is a finite number of points $\alpha_1, \cdots, \alpha_l$ such that when the region t_δ is constructed there remain in it merely a finite number of points $\alpha_1', \cdots, \alpha_{l'}'$, l' not necessarily independent of δ, such that when ϵ and δ', $(0 < \delta' \leqq \delta)$, are chosen at pleasure and $t_{\delta'}$ constructed by removing from t_δ regions of width $2\delta'$ containing respectively the points α_i, there is a length η for which

$$\int_y^{y+\eta} |K(x, \xi)|\, d\xi < \epsilon \quad \begin{cases} x \text{ in } t_{\delta'} \\ (x, y) \text{ and } (x, y+\eta) \text{ in } T. \end{cases}$$

(c') t can be divided into k parts, bounded by points $a = a_0, a_1, \cdots, a_{k-1}, b = a_k$ such that

$$\int_{a_i}^x |K(x, \xi)|\, d\xi \leqq H < 1 \quad \begin{cases} a_i \leqq x \leqq a_{i+1} \\ H \text{ independent of } \delta, \delta'. \end{cases}$$

The conclusion in this case is that there is one and only one solution of the equation continuous in t except in the neighborhood of a finite number of points.

An obvious special case of the theorem of § 1 is where the kernel is continuous throughout T. Another is the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{(x-\xi)^\lambda (x-a)^\mu (\xi-a)^\nu} u(\xi)\, d\xi,$$

where $K(x, \xi)$ is continuous throughout T, and $\lambda + \mu + \nu < 1$.

The condition (A) can be replaced by conditions on the integral of the kernel; for instance (A) and (b) can together be replaced by the condition which follows:

The integral

$$\int_a^x K(x, \xi) r(\xi)\, d\xi$$

where $r(x)$ is finite in t and continuous except for a finite number of points, shall converge except at most for a finite number of values of x, and the function of x thus defined shall remain finite; furthermore, it shall be continuous except at most for a finite number of values of x, denoted by $\alpha_1, \cdots, \alpha_l$, which are independent of the choice of $r(x)$.

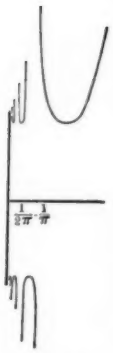

FIG. 6.

A special case of this theorem, elegant on account of the simplicity of the

conditions imposed, has been treated by Dr. W. A. HURWITZ.* The hypotheses for this case were:

(B'') $\phi(x)$ continuous in t;

(a'') $\int_a^t |K(x,\xi)|\,d\xi$ converges in t;

(b'') $\int_a^t |K(x,\xi)|\,d\xi$ represents a continuous function in t;

(c'') $|K(x_1,\xi)| \geqq |K(x_2,\xi)|$ when $x_1 > x_2$.

Here (a'') implies (a) of § 1, and (b'') and (c'') imply (b) and (c). For since

$$|K(x_1,\xi)| \geqq |K(x_2,\xi)|, \qquad x_1 > x_2,$$

$$\int_{a_i}^{x} |K(x,\xi)|\,d\xi \leqq \int_{a_i}^{x} |K(b,\xi)|\,d\xi$$

and

$$\int_{y}^{y+\eta} |K(x,\xi)|\,d\xi \leqq \int_{y}^{y+\eta} |K(b,\xi)|\,d\xi,$$

and it follows from the convergence of

$$\int_a^x |K(b,\xi)|\,d\xi$$

that both (b) and (c) must hold.

§ 6. *Application to a Non-Integrable Kernel.*

An interesting application of the general theorem is to integral equations where the kernels have such singularities that they are no longer integrable. Let us consider the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x,\xi)}{f(\xi)}\,u(\xi)\,d\xi, \tag{9}$$

where $K(x,\xi)$ satisfies Condition (A) and is finite, where $\phi(x)$ satisfies Condition (B) and vanishes at $x = a$ in a way to be explained below, and where $f(\xi)$ is continuous in t and unequal to zero except at the point a, where it may vanish in any way. We assume for convenience that

$$\lim_{y=a} \int_y^x \frac{d\xi}{f(\xi)}$$

* As a problem in Professor BÔCHER's course in integral equations: Harvard University, 1907–08. Dr. HURWITZ also suggests the theorem: If

(1) $K(x,\xi)$ increases with x for any value of ξ;

(2) $\int_a^x |K(x,\xi)|\,d\xi$ converges and represents a continuous function

(3) $\phi(x)$ is finite and integrable;

then the equation has one and only one finite and integrable solution.

does not exist; otherwise the case comes under the theorem of § 1. Without loss of generality we may take

$$|K(x, \xi)| \leqq H < 1,$$

putting whatever multiplicative constant is necessary into the denominator with $f(\xi)$.

If we transform this equation by means of the substitution

$$u(x) = r(x)v(x) \tag{10}$$

we obtain

$$r(x)v(x) = \phi(x) + \int_a^x \frac{K(x, \xi)r(\xi)}{f(\xi)} v(\xi)d\xi,$$

so that

$$v(x) = \frac{u(x)}{r(x)}$$

will be a solution of the equation

$$v(x) = \frac{\phi(x)}{r(x)} + \int_a^x K(x, \xi)\frac{r(\xi)}{f(\xi)r(x)} v(\xi)d\xi; \tag{11}$$

and conversely, if $v(x)$ is a solution of this last equation,

$$u(x) = r(x)v(x)$$

will be the solution of the first.

Now let us see if we can so choose $r(x)$ that the kernel in equation (11) shall satisfy the conditions (a), (b), (c) of the fundamental theorem. That is, $r(\xi)$ will have to be such a function that it will not only erase the singularity due to $f(\xi)$, but will also have enough vanishing left so that by means of the integration it will suffice also for the singularities due to $r(x)$. Obviously, no function with a vanishing point of finite order at a, except in the single case where $f(\xi)$ vanishes to the first order (a case that is taken care of also by the succeeding formulas), will suffice, for the order of vanishing of its integral is greater only by one. And therefore we must look for functions that have essential singularities at the point a. A satisfactory choice is

$$r(x) = e^{-\int_x^b \frac{dx}{|f(x)|}}. \tag{12}$$

With the substitution defined by this value of $r(x)$, the equation for $v(x)$ is

$$v(x) = \phi(x)e^{\int_x^b \frac{dx}{|f(x)|}} + \int_a^x K(x, \xi)\frac{e^{\int_x^b \frac{dx}{|f(x)|}} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}{f(\xi)} d\xi, \tag{13}$$

and the new kernel is

$$\bar{K}(x,\xi) = K(x,\xi)\frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)}. \tag{14}$$

This new kernel $\bar{K}$ satisfies the conditions (a), (b), (c). For the integral

$$\int_a^x |\bar{K}(x,\xi)|\,d\xi$$

converges, $a \leqq x \leqq b$, except for a finite number of points $\lambda_1, \cdots, \lambda_i$ and is $< M$, $x \neq \lambda_1, \cdots, \lambda_i$.

Since $|K(x,\xi)| \leqq H$ we see, when we have defined the points $\alpha_1 \cdots \alpha_i$, that there is a length η such that

$$\int_y^{y+\eta} |\bar{K}(x,\xi)|\,d\xi < \epsilon, \qquad (x,y) \text{ and } (x, y+\eta) \text{ in } \mathcal{I}_\delta.$$

And finally we can show that

$$\int_a^x |\bar{K}(x,\xi)|\,d\xi \leqq H < 1, \quad \begin{cases} a \leqq x \leqq b, \\ x \neq \alpha_1, \cdots, \alpha_i. \end{cases}$$

For

$$\int_a^x |K(x,\xi)| \left| \frac{e^{\int_x^b \frac{d\xi}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} \right| d\xi \leqq H \int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|}\,d\xi.$$

In this integral let

$$\zeta = \int_\xi^b \frac{d\xi}{|f(\xi)|} \qquad \text{and} \qquad z = \int_x^b \frac{dx}{|f(x)|};$$

then

$$d\zeta = -\frac{d\xi}{|f(\xi)|}.$$

When $\xi = a$, we have $\zeta = +\infty$, and when $\xi = x$, we have $\zeta = z$, so that

$$\int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|}\,d\xi$$

becomes

$$\int_z^\infty e^{z-\zeta}\,d\zeta = \left[-e^{z-\zeta}\right]_z,$$

which has the value 1. Whence

$$\int_a^x |\bar{K}(x,\xi)|\,d\xi \leqq H.$$

Hence there is one and only one $v(x)$ which is a solution of (13) finite and continuous in t except for a finite number of points, provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite as x approaches a. Hence by (10) there is at least one solution of the given equation (9), provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite, namely,

$$u(x) = e^{-\int_x^b \frac{dx}{|f(x)|}} v(x). \tag{15}$$

It vanishes at the point a at least as sharply as

$$\text{const. } e^{-\int_x^b \frac{dx}{|f(x)|}}.$$

The case where $f(\xi)$ is discontinuous at a finite number of points is no more general than this that we have treated, for the discontinuities may be taken up by the $K(x, \xi)$. The case where $f(\xi)$ vanishes again in the interval is more complex. If it vanishes as a result of a finite jump, the problem may be treated by the previous analysis if we replace the equation (13) by

$$v(x) = \bar{\phi}(x) + \int_{a_1}^{x} K(x, \xi) \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

where a_1 is this second vanishing point of $f(\xi)$ and where

$$\bar{\phi}(x) = \phi(x)e^{\int_x^{a_1} \frac{dx|}{|f(x)|}} + \int_a^{a_1} K(x, \xi) \frac{e^{\int_x^{a_1} \frac{dx}{|f(x)|} - \int_\xi^{a_1} \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

$v(\xi)$ being known in the interval (a, a_1) as a solution of the equation just studied.

§ 7. *A Generalization of Equation* (9).

An equation closely similar to (9) is

$$u(x) = \phi(x) + \int_a^{x'} \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi) d\xi, \tag{16}$$

where the continuous functions $f(x)$, $g(x)$, which vanish in t only at $x = a$, are such that the limit

$$\lim_{y=a} \int_y^x \frac{d\xi}{f(\xi)g(\xi)}$$

does not exist. If we multiply through by $g(x)$ we get the equation

$$u(x)g(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

and if we put $w(x) = u(x)g(x)$,

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{g(\xi)f(\xi)} w(\xi)d\xi.$$

This equation has a finite solution $w(x)$ continuous in t except for a finite number of points, if

$$\phi(x)g(x)e^{\int_x^b \frac{dx}{|f(x)|\,|g(x)|}}$$

remains finite as x approaches a. Moreover, at the point a, $w(x)$ vanishes as strongly as

$$\text{const} \cdot e^{-\int_x^b \frac{dx}{|f(x)|\,|g(x)|}}.$$

Hence the $u(x)$, which is given by

$$u(x) = \frac{w(x)}{g(x)},$$

vanishes at the point a as strongly as

$$\text{const} \cdot \frac{1}{g(x)} e^{-\int_x^b \frac{dx}{|f(x)|\,|g(x)|}}.$$

§ 8. *A Generalization of Equation* (16).

Finally let us investigate the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)u(\xi)}{g(x)f(\xi)\prod_{i=1}^{p}[\xi - \psi_i(x)]^{\lambda_i}} d\xi, \tag{17}$$

where

$$\sum_{i=1}^{p} \lambda_i = \lambda < 1$$

and the ψ_i are various arbitrary continuous functions.

This equation is equivalent to the equation

$$w(x) = \theta(x) + \int_a^x \frac{\alpha K(x, \xi)}{h(\xi)\prod_{i=1}^{p}[\xi - \psi_i(x)]^{\lambda_i}} w(\xi)d\xi, \tag{18}$$

for (17) is reducible to (18) by the substitutions

$$w(x) = u(x)g(x), \qquad \theta(x) = \phi(x)g(x), \qquad h(x) = \alpha f(x)g(x);$$

and conversely, (18) is reducible to (17) by the inverse substitutions. We shall suppose, as we can without loss of generality, that $|K(x, \xi)| \leqq H < 1$. The α we shall leave undetermined for the present.

By the substitution $w(x) = r(x)v(x)$ the equation (18) is equivalent to

$$v(x) = \frac{\theta(x)}{r(x)} + \int_a^x \frac{r(\xi)}{h(\xi)r(x)} \frac{\alpha K(x, \xi)}{\prod_{i=1}^{p}[\xi - \psi_i(x)]^{\lambda_i}} v(\xi)d\xi. \tag{19}$$

Let us denote by ν any positive number we please less than $1-\lambda$, and choose $r(x)$ so that

$$\int_a^x \left| \left(\frac{r(\xi)}{h(\xi) r(x)} \right)^{\frac{\nu+\lambda}{\nu}} \right| d\xi$$

shall be convergent. Such a choice is

$$r(x) = e^{-\int_x^b \frac{dx}{|h(x)|^{(\nu+\lambda)/\nu}}}.$$

For if we make the substitutions

$$z = \int_x^b \frac{dx}{|h(x)|^{(\nu+\lambda)/\nu}}, \qquad \zeta = \int_\xi^b \frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}}, \qquad d\zeta = -\frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}},$$

the given integral reduces to

$$-(e^z)^{\frac{\nu+\lambda}{\nu}} \int_{\substack{\lim z \\ x=b}}^{z} (e^{-\zeta})^{\frac{\nu+\lambda}{\nu}} d\zeta = (e^z)^{\frac{\nu+\lambda}{\nu}} \frac{\nu}{\nu+\lambda} \int_{\frac{\nu+\lambda}{\nu} z}^{\frac{\nu+\lambda}{\nu} \lim_{x=a} z} e^{-\eta} d\eta = 1 - e^{\frac{\nu+\lambda}{\nu} z} \left(e^{-\frac{\nu+\lambda}{\nu} \lim_{x=a} z} \right).$$

But $z \leqq \lim_{x=a} z$ when $x \geqq a$, so that the second term of this last expression for the integral is less than or equal to 1, and for the given choice of $r(\xi)$ we have

$$\int_a^x \left| \frac{r(\xi)}{h(\xi) r(x)} \right|^{\frac{\nu+\lambda}{\nu}} d\xi < 1.$$

It must be remembered that $r(x)$, by its definition, depends on α.

Let us now consider the integral of the absolute value of the kernel of equation (19) and show that with this definition of $r(x)$ it can be made < 1 throughout t by properly choosing the constant α.

It is obvious that if a, b, m, n are any numbers whatever not negative,

$$a^m b^n \leqq a^{m+n} + b^{m+n};$$

and that if ν is any positive number

$$a^m b^n \leqq a^{m+n/\nu} + b^{m\nu+n}.$$

Moreover if a, m, $b_1, \cdots, b_p$, $n_1, \cdots, n_p$, are any numbers whatever not negative,

$$a^m b_1^{n_1} b_2^{n_2}, \cdots, b_p^{n_p} \leqq a^{m+\frac{n_1+\cdots+n_p}{\nu}} + b_1^{m\nu+n_1+\cdots+n_p} + b_2^{m\nu+n_1+\cdots+n_p} + \cdots + b_p^{m\nu+n_1+\cdots+n_p}.$$

Now

$$\int_a^x \left| \left(\frac{r(\xi)}{h(\xi) r(x)} \right) \frac{\alpha K(x,\xi)}{\prod_{i=1}^{p} \{\xi - \psi_i(x)\}^{\lambda_i}} \right| d\xi \leqq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi) r(x)} \right| \prod_{i=1}^{p} \left| \frac{1}{[\xi - \psi_i(x)]^{\lambda_i}} \right| d\xi$$

$$\leqq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi) r(x)} \right| \prod_{i=1}^{p} \left| \frac{1}{\xi - \psi_i(x)} \right|^{\lambda_i} d\xi.$$

The integrand now, however, is a product of which the factors are all positive. We have then, according to what we have just seen (putting $m = 1$),

$$\left|\frac{r(\xi)}{h(\xi)r(x)}\right| \prod_{i=1}^{p} \left|\frac{1}{\xi - \psi_i(x)}\right|^{\lambda_i} \leqq \left|\frac{r(\xi)}{h(\xi)r(x)}\right|^{1+\lambda/\nu} + \sum_{i=1}^{p} \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda},$$

and the given integral is less than or equal to

$$\alpha H \int_a^x \left|\frac{r(\xi)}{h(\xi)r(x)}\right|^{\frac{\nu+\lambda}{\nu}} d\xi + \sum_{i=1}^{p} \int_a^x \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda} d\xi.$$

Since $\nu + \lambda < 1$ we can choose

$$P \geqq \sum_{i=1}^{p} \int_a^x \left|\frac{1}{\xi - \psi_i(x)}\right|^{\nu+\lambda} d\xi, \qquad x \text{ in } t,$$

so that the integral of the absolute value of the kernel is

$$\leqq \alpha H(1 + P), \qquad x \text{ in } t,$$

the α being involved only explicitly; and by choosing

$$\alpha = \frac{1}{1 + P}$$

this becomes $\leqq H < 1$. There is then under these conditions a solution of equation (19) if $\theta(x)/r(x)$ remains finite.

In regard to (17) we know that there is one solution continuous except at a finite number of points provided that

$$\frac{\phi(x)g(x)}{r(x)}$$

remains finite as x approaches a.

The number of solutions of such equations as these — (9), (16), (17) — we have not investigated. Conceivably there may be, often in fact there are, other finite solutions.

HARVARD UNIVERSITY,
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EIN SEITENSTÜCK ZUR MOEBIUS'SCHEN GEOMETRIE DER KREISVERWANDTSCHAFTEN*

VON

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Öfter wiederholt sich in der Mathematik, und insbesondere in der Geometrie, die Erscheinung, dass Gruppen von Lehrsätzen, die durch gleiche logische Prozesse verbunden sind, in sonst verschiedenen Theorieen auftreten. Es entstehen dann sogenannte Übertragungsprinzipe, für die der von S. LIE gefundene Zusammenhang zwischen Liniengeometrie und Kugelgeometrie das wohl berühmteste Beispiel abgibt. Wir erinnern hier an die Tatsache, dass die von MOEBIUS begründete Art von Kugelgeometrie nicht dem Gedankenkreise der Euklidischen Geometrie (aus dem MOEBIUS sie abgeleitet hat) eigentümlich ist, sondern dass sie in gleicher Weise in den verschiedenen Arten von Nicht-Euklidischer Geometrie auftritt. Für Erscheinungen dieser Art hat man in der Sprache der Gruppentheorie einen präzisen Ausdruck gefunden.

In der Kreisgeometrie von MOEBIUS zum Beispiel wird der Begriff des Kreises derart umgrenzt, dass er auch die geraden Linien umfasst, und es werden dann, bei konsequenter Ausgestaltung der Systematik, ausschliesslich solche Eigenschaften dieser "Kreise" betrachtet, die nicht zerstört werden können durch solche Punkttransformationen, die Kreise in Kreise überführen. Diese Transformationen bilden, wie MOEBIUS selbst schon gefunden hat, eine — nach unserer heutigen Ausdrucksweise — gemischte Gruppe (G_6, H_6), deren allgemeine Transformation von sechs Parametern abhängt.

Die kontinuierliche Gruppe G_6 enthält nun dreigliedrige Untergruppen, die nach LIE's Terminologie *ähnlich* sind zu den ebenfalls dreigliedrigen Bewegungsgruppen der Euklidischen und — je nach Wahl der Raumelemente (Punkt oder geeignetes Punktepaar) — der sphärischen oder der elliptischen, sowie der pseudosphärischen oder der hyperbolischen Geometrie. Danach erscheint es dann als vollkommen selbstverständlich, dass in jeder dieser verschiedenen Arten von Maassgeometrie, also in der Euklidischen Geometrie und in den $2 \cdot 2$ Hauptarten der Nicht-Euklidischen Geometrie an bestimmter Stelle die Lehrsätze von MOEBIUS' Kreisgeometrie auftreten müssen. Aber diese Lehrsätze haben, wiewohl sie unter einander die gleiche Verkettung aufweisen, doch nicht die gleiche

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Bedeutung; sie ordnen sich in verschiedene Gedankensysteme ein; sie können zwar oftmals mit ähnlichen, nicht aber mit den gleichen Worten beschrieben werden, wenigstens nicht, wenn man auf die ursprünglichen Definitionen zurückgeht: Ein *Kreis* in der sphärischen Geometrie zum Beispiel ist nicht Dasselbe, wie die mit dem gleichen Namen bezeichnete Figur der Euklidischen Geometrie. Eben darin ist das Interesse begründet, das solche Wiederholungen derselben Schlussketten haben können, trotzdem sie Wiederholungen sind.

In andern Fällen ist eine so weitgehende logische Aequivalenz nicht vorhanden. Die projektive Geometrie auf einer Kugel zum Beispiel ist sehr verschieden von der Geometrie auf einem Hyperboloïd. Erst, wenn man den elementaren Standpunkt der naïven Betrachtung mit einem andern vertauscht, offenbart sich auch hier eine weitgehende Übereinstimmung, zu deren Verständnis wiederum die Gruppentheorie den Schlüssel liefert. Denkt man sich nämlich die Gruppen der automorphen Kollineationen von Kugel und Hyperboloïd ins komplexe Gebiet erweitert (analytisch fortgesetzt), so sind diese *erweiterten* Gruppen wiederum zu einander *ähnlich* (nämlich zu einander komplex-projektiv).

Daraus ergibt sich, dass alle Lehrsätze der projektiven Geometrie, in denen reelle Figuren keine Sonderstellung einnehmen, von der Kugel auf das Hyperboloïd übertragen werden können, und umgekehrt. Ebenso verhält es sich mit LIE's Kugelgeometrie und der PLÜCKERschen Liniengeometrie. Man hat in solchen Fällen, wie erklärlich, bisher vielleicht zu viel auf das Gemeinsame der verschiedenen Disziplinen, als auf die doch auch vorhandenen und tiefgreifenden Unterschiede geachtet.

In der vorliegenden Untersuchung sollen nun die bekannten Beispiele solchen Zusammenhanges zwischen Theorieen verschiedenen Ursprungs durch ein weiteres vermehrt werden. Die vorzuführende Art von Konstruktionen verhält sich zu den Konstruktionen von MOEBIUS' Kreisgeometrie, die ja von der projektiven Geometrie auf einer Kugelfläche nicht wesentlich verschieden ist, ganz so, wie die Geometrie auf einem Hyperboloïd zur Geometrie auf der Kugel.

Das zugehörige Übertragungsprinzip gehört also zu der zweiten der beiden eben unterschiedenen Arten; erst im komplexen Gebiete fällt unsere Geometrie in logischer Hinsicht zusammen mit der von MOEBIUS; im Reellen sind sie beide sehr verschieden, unbeschadet des Umstandes, dass die tiefer liegenden Analogieen sich natürlich auch hier schon fühlbar machen.

Die *Geometrie der Zykeln*, von der wir reden wollen, entsteht aus der ebenen hyperbolischen Geometrie, wenn man als Raumelement den *Pfeil* betrachtet, nämlich ein wohlgeordnetes Paar von Punkten auf dem gewöhnlich als Kegelschnitt aufgefassten Ort der unendlich fernen Punkte. Diese Pfeile sind das Analogon zu den *Punkten* der MOEBIUSschen Geometrie. Sie werden den Transformationen einer gewissen Gruppe unterworfen, die zu der zuvor genannten MOEBIUS'schen Gruppe ein Seitenstück ist. Die den Kreisen von

MOEBIUS' Geometrie analogen Figuren endlich sind dann eben die weiterhin zu erklärenden Zykeln.

Da es uns nicht nur darauf ankommt, das Gemeinsame unserer Geometrie der Zykeln mit MOEBIUS' Kreisgeometrie hervortreten zu lassen, sondern uns auch daran gelegen ist, unsere Disziplin selbständig zu entwickeln und das Unterscheidende zu seinem Rechte kommen zu lassen, so stellen wir uns von vorneherein auf den Standpunkt der hyperbolischen Geometrie *des reellen Gebietes*. Alle vorkommenden Grössen und Figuren setzen wir als *reell* voraus.

§ 1. *Der Pfeil.*

Das absolute Polarsystem wählen wir in der Form:

$$x_1 y_1 - x_2 y_2 - x_3 y_3 = 0, \tag{1}$$

oder in Linienkoordinaten:

$$\mathfrak{x}_1 \mathfrak{y}_1 - \mathfrak{x}_2 \mathfrak{y}_2 - \mathfrak{x}_3 \mathfrak{y}_3 = 0. \tag{2}$$

Dann lässt sich das binäre Punktgebiet des absoluten Kegelschnitts auf folgende Weise darstellen:

$$x_1 : x_2 : x_3 = \lambda_1^2 + \lambda_2^2 : \lambda_2^2 - \lambda_1^2 : 2\lambda_1\lambda_2. \tag{3}$$

Jedem Werte von $\lambda_1 : \lambda_2$, mit Einschluss des uneigentlichen Wertes ∞, entspricht dann gerade ein unendlich ferner Punkt, und umgekehrt. Als positiv soll die Umlaufsrichtung auf dem absoluten Kegelschnitt bezeichnet werden, in der $\lambda_1 : \lambda_2$ wächst.

Als Raumelement wählen wir den *Pfeil*.* Darunter soll verstanden werden die Figur zweier in eine bestimmte Reihenfolge gesetzter Punkte des absoluten Kegelschnitts. Ein Pfeil hat demnach einen *Anfangspunkt* und einen *Endpunkt*. Liegen beide getrennt, so heisst der Pfeil *eigentlich*. Sein *Träger* ist eine zugängliche Gerade. Sie trägt auch den *entgegengesetzten* Pfeil, d. i. den Pfeil, dessen Anfangspunkt und Endpunkt zusammenfallen mit Endpunkt und Anfangspunkt des ersten.

Fällt der Anfangspunkt eines Pfeils mit seinem Endpunkt zusammen, so reden wir von einem *Punktpfeil*. Ein Punktpfeil ist zu sich selbst entgegengesetzt. Die Gesamtheit der Punktpfeile fungiert mithin als Verzweigungsmannigfaltigkeit. Träger eines Punktpfeils ist eine Minimalgerade (Tangente des absoluten Kegelschnitts).

Eine unzugängliche Gerade trägt keine Pfeile.

Ein Pfeil kann durch ein Paar geordneter Zahlen ($\lambda = \lambda_1 : \lambda_2$, $\mu = \mu_1 : \mu_2$) *dargestellt werden*, wobei (vgl. Gl. 3) $\lambda_1 : \lambda_2$ seinen Anfangspunkt, $\mu_1 : \mu_2$ seinen

* Vgl. analoge Begriffsbildungen bei Hrn. E. STUDY: *Beiträge zur Nicht-Euklidischen Geometrie*; American Journal of Mathematics, vol. 29 (1907); *Über Nicht-Euklidische und Liniengeometrie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 11 (1902), Bd. 15 (1906). Diesen Arbeiten ist die Terminologie grösstenteils nachgebildet.

Endpunkt bedeuten soll. Das System $(\mu_1 : \mu_2, \lambda_1 : \lambda_2)$ gibt dann den entgegengesetzten Pfeil. Verschwindet der Ausdruck

$$(\lambda\mu) = \lambda_1\mu_2 - \lambda_2\mu_1, \tag{4}$$

so liegt ein Punktpfeil vor. Die Zahlen $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$ sollen *Pfeilparameter* genannt werden.

Die Gesamtheit der ∞^2 eigentlichen Pfeile wird durch die ∞^1 Punktpfeile zu einem abgeschlossenen Kontinuum ergänzt.

Zwei Pfeile mit zusammenfallenden Anfangspunkten werden *α-parallel* genannt. Entsprechend heissen Pfeile *ω-parallel*, wenn sie gemeinsame Endpunkte besitzen.

Sind zwei Pfeile zu einem dritten α-parallel (ω-parallel), so sind sie es auch zu einander. Zwei Pfeile, die zugleich α-parallel und ω-parallel sind, fallen zusammen. Zwei Punktpfeile können parallel nur sein, wenn sie zusammen fallen. Durch einen Punkt, der dem absoluten Kegelschnitt nicht angehört, gibt es zu einem Pfeil stets gerade einen α-parallelen und einen ω-parallelen Pfeil.

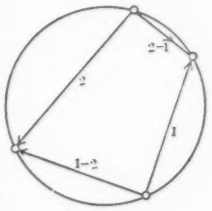


FIG. 1.

Es gibt stets einen einzigen Pfeil, der zu einem ersten Pfeil α-parallel und zu einem zweiten Pfeil ω-parallel ist. Er soll *Verbindungspfeil* (1–2) genannt werden (nicht zu verwechseln mit dem Verbindungspfeil (2–1)). Vgl. Fig. 1.

Die Konstruktion des Verbindens eines ersten Pfeils mit einem zweiten ist stets, *ohne irgend welche Ausnahmen* ausführbar.

Als Koordinaten der geraden Linie $\underline{r}$, die den Pfeil (λ, μ) trägt, findet man:

$$\underline{r}_1 : \underline{r}_2 : \underline{r}_3 = -(\lambda_2\mu_2 + \lambda_1\mu_1) : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1.$$

Setzt man noch:

$$\underline{r}_0 = (\lambda\mu) = \lambda_1\mu_2 - \lambda_2\mu_1, \tag{5}$$

so dürfen die vier Verhältnisgrössen

$$\underline{r}_0 : \underline{r}_1 : \underline{r}_2 : \underline{r}_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : -(\lambda_2\mu_2 + \lambda_1\mu_1) : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1 \tag{6}$$

als *Pfeilkoordinaten* verwandt werden. Zwischen ihnen besteht dann die Relation:

$$\underline{r}_0^2 + \underline{r}_1^2 - \underline{r}_2^2 - \underline{r}_3^2 = 0. \tag{7}$$

Die Systeme $\underline{r}_0 : \underline{r}_1 : \underline{r}_2 : \underline{r}_3$ und $-\underline{r}_0 : \underline{r}_1 : \underline{r}_2 : \underline{r}_3$ ergeben entgegengesetzte Pfeile. Für Punktpfeile hat man: $\underline{r}_0 = 0$.

Durch die *Koordinaten* eines Pfeils sind auch wieder dessen *Parameter* vollkommen bestimmt:

$$\begin{aligned} \lambda_1 : \lambda_2 &= \underline{r}_3 + \underline{r}_0 : \underline{r}_2 - \underline{r}_1 = \underline{r}_1 + \underline{r}_2 : -\underline{r}_3 + \underline{r}_0 \\ \mu_1 : \mu_2 &= \underline{r}_3 - \underline{r}_0 : \underline{r}_2 - \underline{r}_1 = \underline{r}_1 + \underline{r}_2 : -\underline{r}_3 - \underline{r}_0 \end{aligned} \tag{8}$$

Die Gleichung (7) zeigt:

Das Pfeilkontinuum lässt sich ausnahmslos eindeutig umkehrbar abbilden auf die Gesamtheit der Punkte einer singularitätenfreien geradlinigen Fläche zweiter Ordnung.

Diese Fläche, die als einschaliges Hyperboloïd angesehen werden darf, soll kurz *Parasphäre* genannt werden. Entgegengesetzten Pfeilen können zwei Punkte der Parasphäre zugeordnet werden, deren Verbindungsgerade den Mittelpunkt der Fläche trifft. Die Punktpfeile werden dann abgebildet auf den Schnitt der Parasphäre mit der uneigentlichen Ebene des R_3.

Die Pfeilparameter können mit den passend gewählten Parametern der Erzeugenden der Fläche identifiziert werden, sodass sich z. B. entsprechen:

Anfangspunkt eines Pfeils,	Link-seitige Erzeugende;
Endpunkt eines Pfeils,	Recht-seitige Erzeugende;
α-parallele (ω-parallele) Pfeile,	Punkte derselben link-(recht-)seitigen Erzeugenden.

§ 2. *Die parazyklische Gruppe.*

Zu dem erklärten Pfeilkontinuum gehört eine sechsgliedrige Transformationsgruppe, deren reelle Transformationen die Pfeile in der Weise unter einander vertauschen, dass die Anfangspunkte für sich und die Endpunkte für sich *projectiv* unter einander vertauscht werden:

$$(9)\quad \begin{aligned} \lambda_1' &= (\alpha_0+\alpha_3)\lambda_1 + (\alpha_2-\alpha_1)\lambda_2, & \mu_1' &= (\beta_0+\beta_3)\mu_1 + (\beta_2-\beta_1)\mu_2,\\ \lambda_2' &= (\alpha_2+\alpha_1)\lambda_1 + (\alpha_0-\alpha_3)\lambda_2. & \mu_2' &= (\beta_2+\beta_1)\mu_1 + (\beta_0-\beta_3)\mu_2.\\ N(\alpha) &= \alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2 \neq 0, & N(\beta) &= \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 \neq 0. \end{aligned}$$

Wir nennen sie die *parazyklische* Gruppe und bezeichnen sie kurz als G_6. Die genannten reellen Transformationen umfassen vier getrennte Transformationsscharen, die durch folgende Ungleichungen charakterisiert sind:

$$N(\alpha)>0,\ N(\beta)>0;\ N(\alpha)>0,\ N(\beta)<0:$$
$$N(\alpha)<0,\ N(\beta)>0;\ N(\alpha)<0,\ N(\beta)<0.$$

Die erste Schar bildet mit jeder der drei übrigen zusammen eine Gruppe; die von den Transformationen der ersten und vierten Schar $[N(\alpha)\cdot N(\beta)>0]$ gebildete Gruppe werden wir die *engere* Gruppe nennen und mit G_6' bezeichnen.

Eine parazyklische Transformation verwandelt Punktpfeile im Allgemeinen nicht wieder in solche; sie zerstört die Eigenschaft zweier Pfeile, entgegengesetzt zu sein. Dagegen bleibt α-Parallelismus (und ω-Parallelismus) erhalten, denn es gelten die Gleichungen:

$$(\lambda'l') = N(\alpha)\cdot(\lambda l), \qquad (\mu'm') = N(\beta)\cdot(\mu m),$$

wo $(l_1:l_2,\ m_1:m_2)$ einen zweiten Pfeil bedeutet.

Die Gruppe G_6 kann leicht in Pfeil*koordinaten* dargestellt werden; man kann sie folgendermassen schreiben:

$$\mathfrak{x}_i' = c_{i0}\mathfrak{x}_0 + c_{i1}\mathfrak{x}_1 + c_{i2}\mathfrak{x}_2 + c_{i3}\mathfrak{x}_3. \qquad (i = 0, 1, 2, 3) \tag{10}$$

Darin haben die c_{ik} nachstehende Werte:

$$\begin{aligned} c_{00} &= \alpha_0\beta_0 + \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3, \\ c_{11} &= \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3, \\ c_{22} &= \alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3, \\ c_{33} &= \alpha_0\beta_0 - \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3, \end{aligned} \tag{10}$$

$$\begin{aligned} c_{01} &= -\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2, & c_{10} &= \alpha_0\beta_1 - \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2, \\ c_{02} &= -\alpha_0\beta_2 + \alpha_2\beta_0 - \alpha_3\beta_1 + \alpha_1\beta_3, & c_{20} &= -\alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3, \\ c_{03} &= -\alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_1\beta_2 + \alpha_2\beta_1, & c_{30} &= -\alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1. \end{aligned}$$

$$\begin{aligned} c_{23} &= \alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_2\beta_3 - \alpha_3\beta_2, & c_{32} &= -\alpha_0\beta_1 - \alpha_1\beta_0 - \alpha_2\beta_3 - \alpha_3\beta_2, \\ c_{31} &= -\alpha_0\beta_2 - \alpha_2\beta_0 - \alpha_3\beta_1 - \alpha_1\beta_3, & c_{13} &= -\alpha_0\beta_2 - \alpha_2\beta_0 + \alpha_3\beta_1 + \alpha_1\beta_3, \\ c_{12} &= \alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 + \alpha_2\beta_1, & c_{21} &= \alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_1\beta_2 - \alpha_2\beta_1. \end{aligned}$$

Es ist dann:

$$\mathfrak{x}_0'\mathfrak{y}_0' + \mathfrak{x}_1'\mathfrak{y}_1' - \mathfrak{x}_2'\mathfrak{y}_2' - \mathfrak{x}_3'\mathfrak{y}_3' = N(\alpha)\cdot N(\beta)\cdot(\mathfrak{x}_0\mathfrak{y}_0 + \mathfrak{x}_1\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_2 - \mathfrak{x}_3\mathfrak{y}_3),$$

wo die $\mathfrak{y}$ die Koordinaten des Pfeils mit den Parametern $(l_1 : l_2,\ m_1 : m_2)$ sind. Die geometrische Bedeutung des Verschwindens der so gefundenen Simultaninvariante ergibt sich sofort aus der Gleichung:

$$\mathfrak{x}_0\mathfrak{y}_0 + \mathfrak{x}_1\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_2 - \mathfrak{x}_3\mathfrak{y}_3 = 2(\lambda l)(\mu m).$$

Die parazyklische Gruppe lässt sich aber eleganter darstellen. Dabei wird sich zeigen, dass die den Gleichungen (10) anhaftenden Unsymmetrieen nicht vermieden werden können.

Wir bedienen uns eines Systems komplexer Zahlen in vier Einheiten mit folgender Multiplikationstafel:*

$$\begin{array}{cccc} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & e_0 & -e_1 \\ e_3 & e_2 & e_1 & e_0 \end{array} \tag{11}$$

* Das System wird aus dem der Hamiltonschen Quaternionen durch eine lineare nicht reelle Transformation erhalten. Vgl. Enzyklopädie der mathematischen Wissenschaften, I A 4, p. 167, *Zweite Gestalt der Quaternionen.* Das dort unter VI^b angegebene System bedarf nur ganz geringfügiger Abänderung, die hier aber zweckmässig erscheint, um in die Form (11) überzugehen.

Es sei nun:

$$\dot{\alpha} = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \qquad \dot{\beta} = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3.$$

Weiter bedeute $\underline{\dot{\alpha}} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3$ die zu $\dot{\alpha}$ *konjugierte* Quaternion. Dann ist, da e_0 der positiven reellen Einheit gleich gesetzt werden darf:

$$\dot{\alpha}\cdot\underline{\dot{\alpha}} = \underline{\dot{\alpha}}\cdot\dot{\alpha} = N(\alpha)\cdot e_0 = N(\alpha).$$

Ist $\dot{\gamma} = \dot{\alpha}\cdot\dot{\beta}$, so gelten auch die folgenden beiden Gleichungen:

$$\underline{\dot{\gamma}} = \underline{\dot{\beta}}\cdot\underline{\dot{\alpha}}, \qquad N(\gamma) = N(\alpha)\cdot N(\beta).$$

Man darf durch $\dot{\alpha}$ dividieren, solange $N(\alpha) \neq 0$.

Setzen wir noch:

$$\underline{\dot{\mathfrak{x}}} = \underline{\mathfrak{x}}_0 e_0 + \underline{\mathfrak{x}}_1 e_1 - \underline{\mathfrak{x}}_2 e_2 - \underline{\mathfrak{x}}_3 e_3,$$

so stellt sich die parazyklische Gruppe durch die eine Formel dar:

$$\underline{\dot{\mathfrak{x}}}' = \underline{\dot{\alpha}}\cdot\underline{\dot{\mathfrak{x}}}\cdot\dot{\beta}. \tag{12}$$

Die beiden Grössensysteme α_i und β_i $(i = 0, 1, 2, 3)$ sollen *Parameter* der parazyklischen Transformation genannt werden. Jedes dieser Systeme darf, wie die Gleichungen (9) zeigen, noch mit einem beliebigen (nicht verschwindenden) Zahlenfaktor multipliziert werden, ohne dass die dargestellte Transformation sich ändert.

Es ist nun leicht, mehrere parazyklische Transformationen hintereinander auszuführen. Bedeutet nämlich $S_{\alpha\beta}$ die Transformation mit den Parametern α_i, β_i $(i = 0, 1, 2, 3)$, und ist (in WIENERscher Bezeichnung)

$$\underline{\mathfrak{x}}\{S_{\alpha\beta}\}\underline{\mathfrak{x}}'\{S_{\alpha'\beta'}\}\underline{\mathfrak{x}}'' = \underline{\mathfrak{x}}\{S_{\alpha''\beta''}\}\underline{\mathfrak{x}}'',$$

so erhält man:

$$\underline{\dot{\mathfrak{x}}}'' = \underline{\dot{\alpha}}'\cdot\underline{\dot{\mathfrak{x}}}'\cdot\dot{\beta}' = \underline{\dot{\alpha}}'\cdot\underline{\dot{\alpha}}\cdot\underline{\dot{\mathfrak{x}}}\cdot\dot{\beta}\cdot\dot{\beta}' = \underline{\dot{\alpha}}''\cdot\underline{\dot{\mathfrak{x}}}\cdot\dot{\beta}'',$$

wenn die neuen Quaternionen $\dot{\alpha}''$ und $\dot{\beta}''$ folgende Bedeutung haben:

$$\dot{\alpha}'' = \dot{\alpha}\cdot\dot{\alpha}', \quad \dot{\beta}'' = \dot{\beta}\cdot\dot{\beta}'. \tag{13}$$

Die Gleichung:

$$N(\alpha'')\cdot N(\beta'') = N(\alpha)\cdot N(\beta)\cdot N(\alpha')\cdot N(\beta')$$

zeigt, wie die engere Gruppe G_6' zu Stande kommt.

Es gibt stets eine und nur eine parazyklische Transformation, die die Figur dreier wohlgeordneter Pfeile, von denen keine zwei parallel sind, in eine gegebene ebensolche Figur überführt.

§ 3. *Der Zykel.*

Die einfachsten Pfeilmannigfaltigkeiten sind die, deren Gleichungen (in Pfeil*parametern* dargestellt) linear oder bilinear sind.

Es gibt zwei Arten von *linearen* Pfeilmannigfaltigkeiten:

$$(p\lambda) = 0. \qquad (q\mu) = 0.$$

Die Gleichung $(p\lambda) = 0$ stellt ein *Büschel α-paralleler Pfeile* dar, oder wie wir kurz sagen wollen, ein *α-Büschel*. Entsprechend bedeutet die zweite Gleichung ein *ω-Büschel*. Beide Gebilde sind Träger binärer Gebiete.

Jeder Pfeil bestimmt ein einziges α-Büschel und ein einziges ω-Büschel. Ein α-Büschel und ein ω-Büschel durchdringen sich stets in einem einzigen Pfeil. Zwei getrennte α-Büschel (ω-Büschel) haben niemals einen Pfeil gemeinsam.

Die ∞^1 α-Büschel bilden eine einzige Klasse bereits gegenüber den Transformationen einer jeden der vier Scharen, ebenso die ∞^1 ω-Büschel. Ein α-Büschel kann aber durch eine Transformation von G_6 niemals in ein ω-Büschel übergeführt werden (doch vgl. § 4).

Verbindet man einen Pfeil mit sämtlichen Pfeilen eines α-Büschels (ω-Büschels), so entsteht ein α-Büschel (ein einziger Pfeil).

Verbindet man sämtliche Pfeile eines α-Büschels (ω-Büschels) mit einem Pfeil, so entsteht ein einziger Pfeil (ein ω-Büschel). —

Jede *bilineare* Pfeilmannigfaltigkeit ist in der folgenden Form enthalten:

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = 0. \tag{14}$$

Alle durch solche Gleichungen dargestellten Gebilde sollen *Zykeln* genannt werden.

Der Zykel heisst *reduzibel*, sobald seine *Diskriminante*

$$\tfrac{1}{2}D_{aa} = a_{11}a_{22} - a_{12}a_{21} \tag{15}$$

verschwindet. Er besteht dann aus einem α-Büschel und einem ω-Büschel. Er besitzt einen *Mittelpfeil*, den Durchdringungspfeil der beiden Büschel.

Durch irgend drei getrennte Pfeile, von denen gerade zwei parallel sind, geht stets ein und nur ein reduzibler Zykel. Die ∞^2 reduziblen Zykeln bilden eine einzige Klasse gegenüber den Transformationen einer jeden der vier Scharen von G_6.

Zwei getrennte reduzible Zykeln durchdringen sich in zwei getrennten Pfeilen, oder sie haben ein ganzes α-Büschel (ω-Büschel) gemeinsam (*orthogonale* Zykeln, vgl. später § 5).

Liegt ein α-Büschel (ω-Büschel) nicht ganz in einem reduziblen Zykel, so haben beide Figuren stets einen einzigen Pfeil gemeinsam. —

Durch drei getrennte Pfeile, von denen keine zwei α-parallel oder ω-parallel sind, geht stets ein einziger irreduzibler Zykel.

Um mit diesem Begriff gleich eine Anschauung zu verbinden, betrachten wir, indem wir die Ergebnisse späterer Entwicklungen (§ 7) antizipieren, die verschiedenen Gestalten der irreduziblen Zykeln.

Zunächst ist zu nennen der absolute Kegelschnitt (als Ort aller Punktpfeile). Ferner jeder zugängliche Punkt (als Ort aller Pfeile, deren Träger durch ihn hindurchgehen) und jede zugängliche Gerade (als Ort der Pfeile, von deren Trägern sie senkrecht geschnitten wird).

Die übrigen irreduziblen Zykeln lassen sich in der Weise zu Paaren anordnen, dass von den beiden Zykeln eines Paares jeder aus dem anderen hervorgeht, wenn man jeden Zykelpfeil in den entgegengesetzten Pfeil verwandelt. Zu den beiden Zykeln eines Paares gehört demnach ein und dieselbe Geradenmannigfaltigkeit, und diese umhüllt eine irreduzible Kurve 2. Klasse (und 2. Ordnung), die den absoluten Kegelschnitt doppelt berührt (wobei noch die beiden Berührungspunkte zusammenfallen können). Dadurch, dass man diese Kurven (oder ihre im Unendlichen zusammenhängenden Zweige) *orientiert*, kann man die beiden Zykeln des zugehörigen Paares anschaulich unterscheiden.

Doch können von den angeführten Kurven nur diejenigen Zykeln liefern, die überhaupt zugängliche Tangenten besitzen. Indem wir für diese Kurven die in der hyperbolischen Geometrie üblichen Benennungen verwenden, erhalten wir folgende Zusammenstellung:

1. Zugängliche Kreise mit bestimmtem Umlaufssinn (Fig. 2).

2. Zugängliche Abstandskurven (LIEBMANN), *deren Zweige in entgegengesetztem Sinne durchlaufen werden* (Fig. 3).

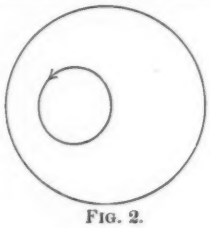


FIG. 2.

FIG. 3.

3. Zugängliche Grenzkurven, deren zugänglicher Zug in bestimmter Richtung durchlaufen wird (Fig. 4).

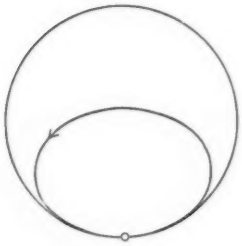


FIG. 4.

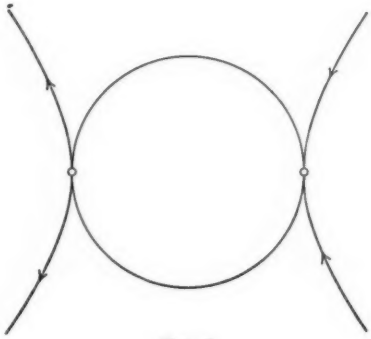


FIG. 5.

4. Unzugängliche Kurven zweiter Ordnung (und zweiter Klasse), die mit dem absoluten Kegelschnitt in *reeller* Doppelberührung liegen, und *deren Tangenten ins zugängliche Gebiet eindringen.* Die Orientierung ist aus der Figur 5 ersichtlich. —

Üben wir auf den durch die Gleichung (14) dargestellten Zykel die parazyklische Transformation $S_{\alpha\beta}$ aus, so reproduziert sich die Diskriminante bis auf den Faktor $N(\alpha) \cdot N(\beta)$:

Die ∞^3 irreduziblen Zykeln bilden eine einzige Klasse gegenüber den Transformationen der Gruppe G_6. Lässt man dagegen nur Transformationen von G_6' zu, so erweist sich das Vorzeichen der Diskriminante als wesentlich, und man hat *zwei* Klassen. Von den *eigentlichen* irreduziblen Zykeln ($D > 0$) sind dann zu unterscheiden die *uneigentlichen* ($D < 0$). Zu den uneigentlichen irreduziblen Zykeln gehören ausser den Geradenzykeln (§ 6) nur noch jene, bei denen die von den Pfeilen umhüllte Kurve zweiter Ordnung und Klasse im unzugänglichen Gebiet liegt (Fig. 5).

Ein α-Büschel (ω-Büschel) und ein irreduzibler Zykel haben stets einen einzigen gemeinsamen Pfeil, und umgekehrt:

*Eine Pfeilmannigfaltigkeit, die mit jedem α-Büschel und jedem ω-Büschel gerade einen Pfeil gemeinsam hat, ist ein irreduzibler Zykel.**

Zwei getrennte irreduzible Zykeln haben zwei, einen oder keinen gemeinsamen Pfeil. Ein reduzibler und ein irreduzibler Zykel durchdringen sich in einem oder in zwei Pfeilen.

Die Gesamtheit der ∞^3 Zykeln lässt sich durchaus eindeutig umkehrbar abbilden auf die Gesamtheit der Ebenen des R_3.

Den reduziblen Zykeln entsprechen dabei die Ebenen einer singularitätenfreien Fläche zweiter Klasse. Diese Fläche kann mit der Parasphäre identifiziert werden. Ferner lässt sich die Abbildung so einrichten, dass sich entspricht:

vereinigte Lage von Pfeil und Zykel,	vereinigte Lage von Bildpunkt auf der Parasphäre mit Bildebene.

Um diesen Gedanken auszuführen, beachten wir, dass sich die Gleichung (14) des Zykels in folgender Gestalt schreiben lässt:

$$[A\underline{r}] = A_0\underline{r}_0 + A_1\underline{r}_1 + A_2\underline{r}_2 + A_3\underline{r}_3 = 0. \tag{16}$$

* Dieser Satz bedarf aber, um nicht falsch verstanden zu werden, einer Erläuterung. In der Tat hat ja jedes orientierte zugängliche Oval, als Ort von Pfeilen aufgefasst, mit jedem α-Büschel (ω-Büschel) gerade einen Pfeil gemeinsam; aber die Aufgabe, die Durchdringungspfeile einer solchen Pfeilmannigfaltigkeit mit einem α-Büschel (ω-Büschel) zu finden, führt auf *nicht-lineare* Gleichungen für die Parameter. Erst wenn man durch Einführung komplexer Pfeile den irreduziblen Zykel zu einer zweifach ausgedehnten Mannigfaltigkeit erweitert, und auch den Begriff des irreduziblen Zykels sachgemäss abändert, würde der Satz im Text ohne weitere Zusätze richtig sein.

Dabei haben die A nachstehende Bedeutung:

$$(17)\quad 2A_0 : 2A_1 : 2A_2 : 2A_3 = a_{12} - a_{21} : -(a_{11} + a_{22}) : a_{22} - a_{11} : a_{12} + a_{21}.$$

Die vier Verhältnisgrössen A_i sollen als *Zykelkoordinaten* verwandt werden. Deuten wir sie als Ebenenkoordinaten im R_3, so ist damit die eben erwähnte Abbildung vollzogen.

Man erkennt leicht die Richtigkeit folgender Gleichung:

$$(18)\quad 2\cdot(A_0B_0 + A_1B_1 - A_2B_2 - A_3B_3) = a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12},$$

wofür wir kürzer schreiben

$$(18)\qquad 2\cdot(AB) = D_{ab}.$$

Eigentlichen (uneigentlichen) irreduziblen Zykeln entsprechen also bei der Abbildung auf die Parasphäre solche Ebenen, deren Pole inbezug auf die Parasphäre innerhalb (ausserhalb)* der Fläche liegen.

Die parazyklische Gruppe geht bei dieser Abbildung über in eine sechsgliedrige projektive Gruppe des R_3, deren Transformationen die Parasphäre in Ruhe lassen. Jetzt erkennt man sofort den tieferen Unterschied zwischen den Transformationen der vier in § 2 erwähnten Scharen. Ist nämlich $N(\alpha)\cdot N(\beta) < 0$, so werden die beiden Gebiete, die wir soeben als Inneres und Äusseres der Parasphäre unterschieden haben, vertauscht.

Für reduzible Zykeln hat man (vgl. Gl. 15)

$$(AA) = \tfrac{1}{2}D_{aa} = 0.$$

§ 4. *Erweiterung der parazyklischen Gruppe.*

Gegenüber parazyklischen Transformationen besitzt die Figur von irgend vier Pfeilen *zwei unabhängige* absolute Invarianten, die wir als α-Doppelverhältnis und ω-Doppelverhältnis bezeichnen:

$$(1234)_\alpha = \frac{(\lambda_1\lambda_3)\cdot(\lambda_4\lambda_2)}{(\lambda_1\lambda_4)\cdot(\lambda_3\lambda_2)}; \qquad (1234)_\omega = \frac{(\mu_1\mu_3)\cdot(\mu_4\mu_2)}{(\mu_1\mu_4)\cdot(\mu_3\mu_2)}.$$

Dabei sind durch die Indizes jetzt nicht Pfeil*parameter*, sondern *Pfeile* unterschieden.

Für irgend vier Pfeile eines α-Büschels (ω-Büschels) ist das α-Doppelverhältnis (ω-Doppelverhältnis) unbestimmt.

Das α-Doppelverhältnis von vier Pfeilen eines irreduziblen Zykels ist gleich ihrem ω-Doppelverhältnis. Umgekehrt:

Ist das α-Doppelverhältnis von vier Pfeilen, von denen keine zwei α-parallel oder ω-parallel sind, gleich ihrem ω-Doppelverhältnis, so gehören die vier Pfeile einem irreduziblen Zykel an.

* Als *innerhalb* bezeichnen wir das Gebiet, in dem der Pol der uneigentlichen Ebene liegt.

Die Gruppe G_6 der parazyklischen Pfeiltransformationen erweitern wir noch durch Hinzufügung der *absoluten Spiegelung*,* d. i. der ausgezeichneten Transformation, die alle Pfeile umkehrt:

$$\begin{gathered}\lambda_1' : \lambda_2' = \mu_1 : \mu_2, \qquad \mu_1' : \mu_2' = \lambda_1 : \lambda_2, \\ \underset{\sim}{\mathfrak{x}}_0' : \underset{\sim}{\mathfrak{x}}_1' : \underset{\sim}{\mathfrak{x}}_2' : \underset{\sim}{\mathfrak{x}}_3' = -\underset{\sim}{\mathfrak{x}}_0 : \underset{\sim}{\mathfrak{x}}_1 : \underset{\sim}{\mathfrak{x}}_2 : \underset{\sim}{\mathfrak{x}}_3, \\ \dot{\underset{\sim}{\mathfrak{x}}}' = \dot{\underset{\sim}{\mathfrak{x}}}.\end{gathered} \tag{19}$$

Aus den parazyklischen Transformationen gehen so hervor die *antiparazyklischen:*

$$\begin{aligned}\lambda_1' &= (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2, & \mu_1' &= (\beta_0 + \beta_3)\lambda_1 + (\beta_2 - \beta_1)\lambda_2, \\ \lambda_2' &= (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2, & \mu_2' &= (\beta_2 + \beta_1)\lambda_1 + (\beta_0 - \beta_3)\lambda_2, \\ N(\alpha) &\neq 0, & N(\beta) &\neq 0.\end{aligned} \tag{20}$$

Sie können vermittelst unserer Quaternionen in die kurze Formel zusammengefasst werden:

$$\dot{\underset{\sim}{\mathfrak{x}}}' = \dot{\underset{\sim}{\alpha}} \cdot \dot{\underset{\sim}{\mathfrak{x}}} \cdot \beta. \tag{21}$$

Die α_i, $\beta_i (i = 0, 1, 2, 3)$ sollen wieder die Parameter der Transformation heissen. Die Zusammensetzung irgend zweier Transformationen der gemischten Gruppe (G_6, H_6) bietet jetzt keine Schwierigkeit:

Es sei $S_{\alpha\beta}$ $(T_{\alpha\beta})$ die parazyklische (antiparazyklische) Transformation mit den Parametern α_i, β_i. Dann hat man (vgl. § 2):

$$\left.\begin{aligned} S_{\alpha\beta} \cdot S_{\alpha'\beta'} &= S_{\alpha''\beta''} \\ T_{\alpha\beta} \cdot S_{\alpha'\beta'} &= T_{\alpha''\beta''} \end{aligned}\right\} \dot{\alpha}'' = \dot{\alpha} \cdot \dot{\alpha}', \quad \dot{\beta}'' = \dot{\beta} \cdot \dot{\beta}',$$

$$\left.\begin{aligned} S_{\alpha\beta} \cdot T_{\alpha'\beta'} &= T_{\alpha''\beta''} \\ T_{\alpha\beta} \cdot T_{\alpha'\beta'} &= S_{\alpha''\beta''} \end{aligned}\right\} \dot{\alpha}'' = \dot{\beta} \cdot \dot{\alpha}', \quad \dot{\beta}'' = \dot{\alpha} \cdot \dot{\beta}'. \tag{22}$$

Auch die Gesamtheit H_6 der antiparazyklischen Transformationen zerfällt in vier getrennte Scharen, die entsprechend, wie die von G_6 charakterisiert sind. Die Gruppe (G_6, H_6) enthält wieder eine sechsgliedrige Untergruppe (G_6', H_6'), charakterisiert durch die Ungleichung

$$N(\alpha) \cdot N(\beta) > 0.$$

Eine antiparazyklische Transformation vertauscht α-Parallelismus und ω-Parallelismus; sie vertauscht ferner das α-Doppelverhältnis mit dem ω-Doppelverhältnis.

Die Gruppe (G_6, H_6) umfasst alle Transformationen von Pfeilen, die Büschel paralleler Pfeile in ebensolche, und Zykeln in Zykeln überführen.

* Der Name wird weiterhin (§ 5) motiviert werden.

Die letzte Hälfte dieses Satzes kann auch folgendermassen präzisiert werden:

Es gibt stets eine einzige *antiparazyklische* Transformation, die die Figur dreier wohlge ordneter Pfeile, von denen keine zwei parallel sind, in eine vorgegebene Figur derselben Eigenschaft überführt.

Gegenüber der Gruppe (G_6, H_6) gibt es nur noch *eine* Klasse von Büscheln paralleler Pfeile (vgl. § 3).

Bei der am Schlusse des vorigen Abschnitts erwähnten Abbildung der Zykeln auf den Ebenenraum geht die Gruppe (G_6, H_6) über in die Gruppe *aller* projektiven Transformationen der R_3, die die Parasphäre in Ruhe lassen, (G_6', H_6') in die Gruppe *aller* projektiven Transformationen des R_3, die ausserdem die beiden Seiten der Parasphäre nicht vertauschen.

§ 5. *Spiegelung am Zykel.*

Mit jedem *irreduziblen* Zykel A ist eine *involutorische antiparazyklische* Pfeiltransformation verbunden, die wir als *Spiegelung am Zykel A* bezeichnen:

$$(23)\quad \begin{aligned} \lambda_1' &= (A_0 - A_3)\mu_1 + (A_1 - A_2)\mu_2, \quad \mu_1' = (A_0 + A_3)\lambda_1 + (A_2 - A_1)\lambda_2, \\ \lambda_2' &= -(A_1 + A_2)\mu_1 + (A_0 + A_3)\mu_2, \quad \mu_2' = (A_2 + A_1)\lambda_1 + (A_0 - A_3)\lambda_2. \end{aligned}$$

Von zwei durch diese Transformation zugeordneten Pfeilen sagen wir: jeder liegt *invers* zum andern inbezug auf den Zykel A.

Liegen zwei Pfeile invers inbezug auf den irreduziblen Zykel A, so gehören ihre beiden Verbindungspfeile dem Zykel an.

Dieser Satz liefert eine höchst einfache Lösung der Aufgabe:

Einen Pfeil (λ, μ) *an einem irreduziblen Zykel A zu spiegeln.* (Fig. 6.)

Man zieht die beiden Zykelpfeile, die dem gegebenen parallel sind. Sie ergeben gerade noch einen Verbindungspfeil; dieser ist der gesuchte zu (λ, μ) inbezug auf A invers liegende (λ', μ'). Die Konstruktion versagt niemals.

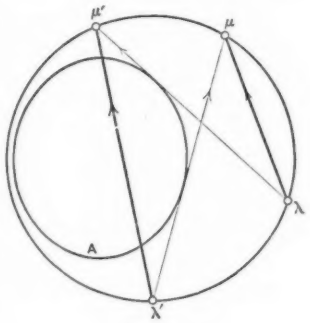


FIG. 6.

Jeder Pfeil liegt zu dem entgegengesetzten Pfeil invers inbezug auf den Zykel der Punktpfeile. Dieser wird später als *absoluter* Zykel bezeichnet werden, und daraus motiviert sich die Bezeichnung absolute Spiegelung (§ 4) für die ausgezeichnete Transformation, die alle Pfeile umkehrt.

Die Pfeile des spiegelnden Zykels selbst, und nur diese, sind ihre eigenen Spiegelbilder.

Das Spiegelbild eines α-Büschels ist ein ω-Büschel, das eines irreduziblen

(reduziblen) Zykels wieder ein irreduzibler (reduzibler) Zykel. Eigentliche (uneigentliche) Zykeln werden wieder in eigentliche (uneigentliche) übergeführt.

Bilden wir die Gleichungen (23) für einen *reduziblen* Zykel, so verschwinden die Transformationsdeterminanten. Von Spiegelungen an reduziblen Zykeln reden wir daher nicht.

Durch Zusammensetzung einer geraden (ungeraden) Anzahl von Spiegelungen an Zykeln erhält man alle parazyklischen (antiparazyklischen) Transformationen *der engeren Gruppe* (G_6', H_6').

Die Transformationen von (G_6, H_6), die der engeren Gruppe (G_6', H_6') nicht angehören, können nicht durch involutorische parazyklische oder antiparazyklische Pfeiltransformationen erhalten werden.

Wir stellen die Spiegelung am Zykel A nunmehr auch in *Pfeilkoordinaten* auf.

$$
\begin{aligned}
-\underset{\sim}{\mathfrak{x}}_0' &= \quad 2\cdot A_0\cdot[A\underset{\sim}{\mathfrak{x}}] - (AA)\underset{\sim}{\mathfrak{x}}_0,\\
-\underset{\sim}{\mathfrak{x}}_1' &= \quad 2\cdot A_1\cdot[A\underset{\sim}{\mathfrak{x}}] - (AA)\underset{\sim}{\mathfrak{x}}_1,\\
-\underset{\sim}{\mathfrak{x}}_2' &= -2\cdot A_2\cdot[A\underset{\sim}{\mathfrak{x}}] - (AA)\underset{\sim}{\mathfrak{x}}_2,\\
-\underset{\sim}{\mathfrak{x}}_3' &= -2\cdot A_3\cdot[A\underset{\sim}{\mathfrak{x}}] - (AA)\underset{\sim}{\mathfrak{x}}_3.
\end{aligned}
\tag{24}
$$

Bei Benutzung von *Zykel*koordinaten hat man:

$$
\begin{gathered}
X_i' = (AA)X_i - 2(AX)A_i, \qquad (i=0,1,2,3),\\
(X'X') = (AA)^2\cdot(XX), \qquad (X'Y') = (AA)^2\cdot(XY).
\end{gathered}
\tag{25}
$$

Bei der Spiegelung an einem Zykel bleiben (ausser ihm selbst) ∞^2 Zykeln einzeln in Ruhe. Diese sollen zum spiegelnden Zykel *orthogonal* genannt werden. Ist ein irreduzibler Zykel Y orthogonal zu dem irreduziblen Zykel X, so ist auch X orthogonal zu Y. Diese Beziehung wird ausgedrückt durch eine symmetrische Gleichung (vgl. Gl. 18)

$$(XY) = 0.$$

Diese Gleichung *definiert* die orthogonale Lage eines irreduziblen (oder reduziblen) Zykels zu einem *reduziblen*, wo die obige Erklärung versagt. Aus der Symmetrie dieser Gleichung folgt, dass man jetzt allgemein von *zu einander* orthogonalen Zykeln reden darf. Jetzt wird auch die bereits in § 3 gebrauchte Bezeichnung orthogonale Lage zweier reduzibler Zykeln gegeneinander motiviert; insonderheit darf jeder reduzible Zykel orthogonal zu sich selbst genannt werden.

Orthogonale Lage zweier Zykeln bleibt bei jeder parazyklischen und jeder antiparazyklischen Transformation erhalten, d. i. auch dann, wenn die Transformation nicht in der Gruppe (G_6', H_6') enthalten ist.

Die hier aufgetretenen Analogieen mit der Lehre von den MOEBIUS'schen

Kreisverwandtschaften erstrecken sich noch auf eine Reihe von weiteren Sätzen, bei deren Aufzählung wir uns kurz fassen werden. Nur abweichendes Verhalten soll besonders betont werden.

Die Figur aller ∞^2 Zykeln, die zu einem Zykel orthogonal liegen, bildet ein *Zykelbündel*. Der Zykel, zu dem alle Zykeln des Bündels orthogonal liegen, heisse *Mittelzykel* des Zykelbündels; er gehört dem Zykelbündel aber nur dann an, wenn er *reduzibel* ist. Wir klassifizieren die Zykelbündel:

1. Mittelzykel irreduzibel ∞^3. Eine Klasse.
   - *a.* Mittelzykel eigentlich ∞^3. Eine Klasse.
   - *b.* Mittelzykel uneigentlich ∞^3. Eine Klasse.
2. Mittelzykel reduzibel ∞^2. Eine Klasse.

Die Einteilung (1, 2, ···) soll sich (wie auch bei der Klassifikation der Zykel*büschel*) auf die Gruppe (G_6, H_6) beziehen, während die Einteilung (a, b) sich ergibt, wenn man nur Transformationen von (G_6', H_6') zulässt. Es gibt also gegenüber (G_6, H_6) *zwei*, gegenüber (G_6', H_6') *drei* Klassen von Zykelbündeln.

Zwei getrennte Zykelbündel durchdringen einander in einem *Zykelbüschel*. Diese werden klassifiziert durch Betrachtung der in ihnen enthaltenen reduziblen Zykeln.

1. Kein reduzibler Zykel. ∞^4. Eine Klasse gegenüber (G_6, H_6), zwei Klassen gegenüber (G_6', H_6'):
   - *a.* Das Büschel besteht ganz aus eigentlichen Zykeln. ∞^4. Eine Klasse.
   - *b.* Das Büschel besteht ganz aus uneigentlichen Zykeln ∞^4. Eine Klasse.
2. Zwei getrennte reduzible Zykeln. ∞^4. Eine Klasse.
3. Ein (doppelt zählender) reduzibler Zykel. ∞^3. Eine Klasse gegenüber (G_6, H_6), zwei Klassen gegenüber (G_6', H_6'):
   - *a.* Alle irreduziblen Zykeln sind eigentlich ∞^3. Eine Klasse.
   - *b.* Alle irreduziblen Zykeln sind uneigentlich ∞^3. Eine Klasse.
4. Das Büschel besteht ganz aus reduziblen Zykeln. ∞^1. Eine Klasse.

Jedem Zykelbüschel ist ein zweites *reziprok* zugeordnet:

In reziproken Zykelbüscheln liegt jeder Zykel des einen Büschels orthogonal zu sämtlichen Zykeln des anderen Büschels. Reziproke Zykelbüschel gehören demselben Haupttypus (aber in den Fällen 1. und 3. getrennten Untertypen) an. Jedes Büschel des Typus 4. ist zu sich selbst reziprok.

Bildet man (§ 3) die Zykeln auf die Ebenen des R_3 ab, so erscheinen die Zykelbündel als Punkte, die Zykelbüschel als gerade Linien.

§ 6. *Die Bewegungen und Umlegungen.*

Wir gehen jetzt zur *elementaren* Pfeilgeometrie über. Der Zykel der Punktpfeile wird als *absoluter Zykel* erklärt und es werden nur solche Transformationen von (G_6, H_6) zugelassen, die den absoluten Zykel in sich selbst

überführen. Diese werden dann *Bewegungen* oder *Umlegungen* genannt, jenachdem sie parazyklisch oder antiparazyklisch sind.

Eine Transformation von G_6 (H_6) ist eine Bewegung (Umlegung), wenn zwischen ihren Parametern die Gleichungen bestehen:

$$\beta_0 : \beta_1 : \beta_2 : \beta_3 = \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 .$$

Die Bewegungen und Umlegungen sind also bereits in der Gruppe (G_6', H_6') enthalten.

Die Bewegungen und Umlegungen verwandeln entgegengesetzte Pfeile stets wieder in ebensolche. Sie umfassen *alle* Transformationen von (G_6, H_6), die diese Eigenschaft besitzen.

Nennen wir *antiparallel* zwei Pfeile, wenn der Endpunkt des einen mit dem Anfangspunkt des andern zusammenfällt, so gilt der Satz:

Antiparallelismus von Pfeilen bleibt bei jeder Bewegung und Umlegung erhalten.

Die Bewegungen und Umlegungen umfassen sämtliche Transformationen von (G_6, H_6), die antiparallele Pfeile stets wieder in ebensolche verwandeln.

Man kann die Bewegungen und Umlegungen durch die Quaternionengleichungen darstellen:

$$\underset{\sim}{\dot{x}}' = \underset{\sim}{\dot{\alpha}} \cdot \underset{\sim}{\dot{x}} \cdot \dot{\alpha}, \qquad \underset{\sim}{\dot{x}}' = \underset{\sim}{\dot{\alpha}} \cdot \underset{\sim}{\dot{x}} \cdot \dot{\alpha}. \tag{26}$$

Die Gleichungen (10) vereinfachen sich wesentlich:

$$\begin{aligned}
c_{01} = c_{02} = c_{03} = 0, \qquad & c_{10} = c_{20} = c_{30} = 0, \\
c_{00} &= \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2, \\
c_{11} &= \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\
c_{22} &= \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2, \\
c_{33} &= \alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2, \\
c_{23} = 2(\alpha_0\alpha_1 - \alpha_2\alpha_3), \qquad & c_{32} = 2(-\alpha_0\alpha_1 - \alpha_2\alpha_3), \\
c_{31} = 2(-\alpha_0\alpha_2 - \alpha_3\alpha_1), \qquad & c_{13} = 2(-\alpha_0\alpha_2 + \alpha_3\alpha_1), \\
c_{12} = 2(\alpha_0\alpha_3 + \alpha_1\alpha_2), \qquad & c_{21} = 2(\alpha_0\alpha_3 - \alpha_1\alpha_2).
\end{aligned} \tag{27}$$

Übt man nach der Bewegung (Umlegung) $\dot{\alpha}$ die Bewegung oder Umlegung $\dot{\alpha}'$ aus, so gilt jetzt für die resultierende Transformation $\dot{\alpha}''$ *in allen Fällen* die Gleichung:

$$\dot{\alpha}'' = \dot{\alpha} \cdot \dot{\alpha}',$$

oder ausgeschrieben:

$$\begin{aligned}
\alpha_0'' &= \alpha_0\alpha_0' - \alpha_1\alpha_1' + \alpha_2\alpha_2' + \alpha_3\alpha_3', \\
\alpha_1'' &= \alpha_1\alpha_0' + \alpha_0\alpha_1' + \alpha_3\alpha_2' - \alpha_2\alpha_3', \\
\alpha_2'' &= \alpha_2\alpha_0' + \alpha_3\alpha_1' + \alpha_0\alpha_2' - \alpha_1\alpha_3', \\
\alpha_3'' &= \alpha_3\alpha_0' - \alpha_2\alpha_1' + \alpha_1\alpha_2' + \alpha_0\alpha_3'.
\end{aligned} \tag{28}$$

Ferner ist:

$$N(\alpha'') = N(\alpha) \cdot N(\alpha').$$

Aus dieser Gleichung schliessen wir:

Die Gesamtheit der Bewegungen zerfällt in zwei getrennte Schaaren:

1) $N(\alpha) > 0$. *Eigentliche* Bewegungen.

2) $N(\alpha) < 0$. *Uneigentliche* Bewegungen.

Die *uneigentlichen* Bewegungen kehren den Umlaufssinn auf dem absoluten Kegelschnitt, d. i. dem als Punktmannigfaltigkeit aufgefassten absoluten Zykel, um.

Die *Gruppe* der *eigentlichen* Bewegungen ist kontinuierlich. Der Umlaufssinn bleibt ungeändert.

Entsprechend haben wir bei den Umlegungen zu unterscheiden. *Eigentliche* Umlegungen lassen den Umlaufssinn ungestört $[N(\alpha) > 0]$.

Es bezeichne S (T) eine eigentliche, S' (T') eine uneigentliche Bewegung (Umlegung).

Drückt man dann durch die symbolische Gleichung

$$(SS) = S$$

den Tatbestand aus, dass zwei eigentliche Bewegungen wieder eine eigentliche Bewegung ergeben, so gelten auch folgende Gleichungen:

$$\begin{aligned} (SS) = S, \quad (SS') = S', \quad (ST) = T, \quad (ST') = T'; \\ (S'S') = S, \quad (S'T) = T', \quad (S'T') = T; \\ (TT) = S, \quad (TT') = S'; \\ (T'T') = S. \end{aligned}$$

Es gibt also folgende dreigliedrige reell-gemischte Gruppen:

$$\overline{S, S', T, T'}, \quad \overline{S, T}, \quad \underline{S, T'}, \quad \overline{S, S'}.$$

Bei einer Bewegung (Umlegung) werden die Anfangspunkte der Pfeile ebenso transformiert, wie die Endpunkte. Die Bewegungen und Umlegungen sind also auch Transformationen der Punkte des absoluten Kegelschnitts, d. i. des als Punktmannigfaltigkeit aufgefassten absoluten Zykels.

Betrachtet man bei dieser Auffassung jeden ursprünglichen unendlich fernen Punkt als Anfangspunkt, jeden transformierten als Endpunkt eines Pfeils, so entsteht eine Mannigfaltigkeit von ∞^1 Pfeilen, die ein *irreduzibler Zykel* sein muss, da sie offenbar mit jedem α-Büschel und mit jedem ω-Büschel gerade einen Pfeil gemeinsam hat.

Der Bewegung $\dot{\alpha}$ ist dabei derselbe Zykel zugeordnet, wie der Umlegung $\dot{\alpha}$; er hat die Koordinaten $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$, also die Gleichung $\alpha_0 \mathfrak{x}_0 + \alpha_1 \mathfrak{x}_1 + \alpha_2 \mathfrak{x}_2 + \alpha_3 \mathfrak{x}_3 = 0$.

Die identische Bewegung, ebenso demnach die absolute Spiegelung, führt auf den absoluten Zykel $1:0:0:0$.

Die Theorie der Zykeln steht also in engem Zusammenhang mit der Lehre von den Bewegungen.

Wir betrachten jetzt die *involutorischen* Bewegungen und Umlegungen. Für sie muss, wenn wir von der identischen Bewegung und der absoluten Spiegelung absehen, der Parameter α_0 verschwinden.

1. *Die eigentlichen involutorischen Bewegungen.* $N(\alpha) > 0$. Kein Pfeil bleibt in Ruhe, dagegen werden ∞^1 Pfeile umgekehrt. Die Träger aller dieser Pfeile laufen durch den (zugänglichen) Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

Umwendung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.*

Der zugehörige Zykel besteht aus den erwähnten ∞^1 Pfeilen und soll *Punktzykel* genannt werden. Er besitzt keine Punktpfeile und gestattet die absolute Spiegelung. Er gehört zu den *eigentlichen* Zykeln (§ 3).

2. *Die uneigentlichen involutorischen Bewegungen.* $N(\alpha) < 0$. In Ruhe bleiben zwei entgegengesetzte Pfeile und damit auch die zugehörigen Verbindungs(Punkt-)pfeile. Auch hier werden ∞^1 Pfeile umgekehrt. Ihre Träger schneiden jetzt den Träger $\alpha_1 : -\alpha_2 : -\alpha_3$ des ersten Pfeilpaares senkrecht.

Umwendung um die Gerade $\alpha_1 : -\alpha_2 : -\alpha_3$.

Der zugehörige aus jenen ∞^1 Pfeilen und zwei Punktpfeilen bestehende Zykel soll *Geradenzykel* genannt werden. Er gestattet die absolute Spiegelung.

3. *Die eigentlichen involutorischen Umlegungen.* $N(\alpha) > 0$. Der (*eigentliche*) Punktzykel mit den Koordinaten $0 : \alpha_1 : \alpha_2 : \alpha_3$ bleibt *pfeilweise* in Ruhe.

Spiegelung am Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

4. *Die uneigentlichen involutorischen Umlegungen.* $N(\alpha) < 0$. Der (*uneigentliche*) Geradenzykel mit den Koordinaten $0 : \alpha_1 : \alpha_2 : \alpha_3$ bleibt *pfeilweise* in Ruhe.

Spiegelung an der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Die Spiegelungen an Punkten und Geraden fallen unter den allgemeineren Begriff: Spiegelung am Zykel. Damit ist also die Frage nach der Konstruktion dieser Transformationen bereits beantwortet. Entsprechendes gilt auch für die Konstruktion der Umwendungen.

§ 7. *Klassifikation der Bewegungen und der Zykeln.*

Jede von der identischen verschiedene Bewegung lässt sich auf ∞^1 *Arten darstellen als Aufeinanderfolge zweier Spiegelungen.*

* Statt *Umwendung um einen Punkt* pflegt man auch zu sagen: *Spiegelung am Punkt.* Wir sind hier aber gezwungen, eine Trennung in der Bezeichnung vorzunehmen, und verwenden das Wort Umwendung nur für Bewegungen, das Wort Spiegelung nur für Umlegungen.

Es sei

$$S_\alpha = T_\beta \cdot T_\gamma$$

Dann darf, wie man auf Grund der Gll. (28) leicht bestätigt, gesetzt werden:

$$\beta_0 : \beta_1 : \beta_2 : \beta_3 = 0 : \sigma_2\alpha_3 - \sigma_3\alpha_2 : -(\sigma_3\alpha_1 - \sigma_1\alpha_3) : -(\sigma_1\alpha_2 - \sigma_2\alpha_1),$$

$$\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 = 0 : \tau_2\alpha_3 - \tau_3\alpha_2 : -(\tau_3\alpha_1 - \tau_1\alpha_3) : -(\tau_1\alpha_2 - \tau_2\alpha_1).$$

Dabei sind die σ und τ durch folgende Relationen verbunden:

$$\left.\begin{aligned} \tau_1 &= \alpha_0\sigma_1 - \alpha_3\sigma_2 + \alpha_2\sigma_3, \\ \tau_2 &= -\alpha_3\sigma_1 + \alpha_0\sigma_2 + \alpha_1\sigma_3, \\ \tau_3 &= \alpha_2\sigma_1 - \alpha_1\sigma_2 + \alpha_0\sigma_3, \end{aligned}\right\} (\sigma_1 : \sigma_2 : \sigma_3 \neq \alpha_1 : \alpha_2 : \alpha_3).$$

Ferner ist

$$N(\alpha) = N(\beta) \cdot N(\gamma).$$

Eine *eigentliche* Bewegung kann zusammengesetzt werden aus zwei Spiegelungen an Punkten oder auch aus zwei Spiegelungen an Geraden.

Um eine *uneigentliche* Bewegung zu erhalten, hat man die Spiegelung an einem Punkt zusammenzusetzen mit der Spiegelung an einer Geraden.

Wir untersuchen jetzt die verschiedenen Fälle, indem wir mit den eigentlichen Bewegungen beginnen.

1. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 > 0$. Die beiden Geraden $\beta_1 : -\beta_2 : -\beta_3$ und $\gamma_1 : -\gamma_2 : -\gamma_3$ laufen durch den Punkt $\alpha_1 : \alpha_2 : \alpha_3$. Kein Pfeil bleibt in Ruhe, ebenso kein reduzibler Zykel, wohl aber der Punktzykel $0 : \alpha_1 : \alpha_2 : \alpha_3$.

Drehung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

Besonderer Fall: $\alpha_0 = 0$. Umwendung um den Punkt $\alpha_1 : \alpha_2 : \alpha_3$.

2. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 0$. Die beiden vorhin erwähnten Geraden sind jetzt parallel. In Ruhe bleibt ein Punktpfeil und ein reduziber Zykel, der die absolute Spiegelung gestattet.

Gleitung längs des Punktpfeils* $0 : \alpha_1 : -\alpha_2 : -\alpha_3$.

3. $\alpha_1^2 - \alpha_2^2 - \alpha_3^2 < 0$. Jetzt stellt das System $\alpha_1 : -\alpha_2 : -\alpha_3$ eine zugängliche Gerade dar, die Verbindungslinie der Punkte $\beta_1 : \beta_2 : \beta_3$ und $\gamma_1 : \gamma_2 : \gamma_3$, oder gemeinsame Normale der Geraden $\beta_1 : -\beta_2 : -\beta_3$ und $\gamma_1 : -\gamma_2 : -\gamma_3$. In Ruhe bleiben die beiden Pfeile auf der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$, sowie die zugehörigen Verbindungspfeile, ferner vier unschwer zu beschreibende reduzible Zykeln, endlich der Geradenzykel $0 : \alpha_1 : \alpha_2 : \alpha_3$.

Gleitung längs der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Ist die Bewegung S_α uneigentlich, so sind die Mittelelemente der Spiegelungen T_β und T_γ Punkt auf der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$ und Gerade senkrecht

* Von Hrn. E. STUDY benutzt. Das sonst übliche Wort *Schiebung* wird besser vermieden, da es Analogieen suggeriert, die nicht existieren.

zu ihr. Es bleiben dieselben Gebilde in Ruhe, wie bei der eben besprochenen Gleitung.

Uneigentliche Gleitung längs der Geraden $\alpha_1 : -\alpha_2 : -\alpha_3$.

Besonderer Fall: $\alpha_0 = 0$. Unwendung um die Gerade $\alpha_1 : -\alpha_2 : -\alpha_3$.

In allen Fällen ist die Maassinvariante (Winkel, Öffnung*, Abstand) der Mittelelemente der beiden Spiegelungen gleich der halben Grösse der resultierenden Bewegung (Drehwinkel, Gleitöffnung,† Gleitstrecke).

Überall in diesem Abschnitt darf das Wort Spiegelung ersetzt werden durch Umwendung.

Wir geben jetzt im Zusammenhang die Parameter einer Bewegung von vorgeschriebenen üblichen Bestimmungsstücken an.

1. Drehung um den Punkt x durch den Winkel 2ϑ:

$$-\sqrt{x_1^2 - x_2^2 - x_3^2}\cot\vartheta : x_1 : x_2 : x_3.$$

2. Gleitung längs der Minimalgeraden $\underset{\sim}{r}$ durch die Öffnung 2ω:

$$-\frac{1}{\omega}\underset{\sim}{r}_1 : \underset{\sim}{r}_1 : -\underset{\sim}{r}_2 : -\underset{\sim}{r}_3.\dagger$$

3. Gleitung längs des (eigentlichen) Pfeils $\underset{\sim}{r}$ durch die Strecke 2η:

$$\underset{\sim}{r}_0 \coth\eta : \underset{\sim}{r}_1 : -\underset{\sim}{r}_2 : -\underset{\sim}{r}_3.$$

4. Uneigentliche Gleitung längs des Pfeils $\underset{\sim}{r}$ durch die Strecke 2η:

$$\underset{\sim}{r}_0 \operatorname{tgh}\eta : \underset{\sim}{r}_1 : -\underset{\sim}{r}_2 : -\underset{\sim}{r}_3.$$

---

* Vgl. über diese Begriffe weiter unter die folgenden beiden Fussnoten.

† Hierzu bemerken wir noch Folgendes: Eine *invariante* Grösse, die durch zwei Punkte einer Minimalgeraden (oder, was dasselbe ist, durch zwei parallele Gerade) bestimmt wäre, gibt es natürlich nicht. Erklärt man die Öffnung zweier paralleler Gerader $\underset{\sim}{r}$ und $\underset{\sim}{r}'$ durch die Gleichung

$$-2\omega = [\underset{\sim}{r}, \underset{\sim}{r}'] = \frac{\underset{\sim}{r}_2\underset{\sim}{r}'_3 - \underset{\sim}{r}_3\underset{\sim}{r}'_2}{\underset{\sim}{r}_1\underset{\sim}{r}'_1 - \underset{\sim}{r}_2\underset{\sim}{r}'_2 - \underset{\sim}{r}_3\underset{\sim}{r}'_3},$$

so ist zu beachten, dass diese Grösse invariant ist nur gegenüber den Bewegungen der im Texte unter (2) aufgeführten eingliedrigen Gruppe, die also nur einen einzigen unendlich fernen Punkt in Ruhe lassen, nämlich den unendlich fernen Schnittpunkt der beiden Parallelen. Sofern die Figur zweier getrennten parallelen Geraden, die in eine bestimmte Reihenfolge gesetzt sind, eine Bewegung der soeben beschriebenen Art völlig bestimmt, darf man von der Öffnung dieser Bewegung reden.

Invariant ist aber die *Öffnungshalbierende* zweier paralleler Geraden, eine dritte zugängliche parallele Gerade, die Symmetrieaxe der Figur der beiden ersten ist.

Will man den Begriff der Öffnung vermeiden, so empfiehlt sich etwa folgende Fragestellung: *Wie heissen die Parameter der Gleitung längs der Minimalgeraden* $L_1 : L_2$, *die die Minimalgerade* $M_1 : M_2$ *überführt in die Minimalgerade* $N_1 : N_2$? Man findet:

$$\frac{2(LM)(LN)}{(MN)} : L_2^2 + L_1^2 : L_2^2 - L_1^2 : 2L_1L_2,$$

wo $L_1 : L_2$ den Berührungspunkt der ersten Minimalgeraden darstellt (vgl. § 1, Gl. 3) und das Symbol (MN) die folgende Bedeutung hat:

$$(MN) = M_1N_2 - M_2N_1, \text{ usw.}$$

Mit der Einteilung der Bewegungen ist gleichzeitig bereits die Klassifikation der irreduziblen Zykeln (gegenüber Bewegungen und Umlegungen) vollzogen. Wir stellen kanonische Vertreter für die einzelnen Typen auf.

I. *Irreduzible Zykeln.* $(AA) \neq 0$.

a. Eigentliche Zykeln. $(AA) > 0$.

1. $A_1^2 - A_2^2 - A_3^2 > 0$.

α) $A_0 \neq 0$. *Drehzykel.* ∞^3. ∞^1 Klassen.

$$\cos\vartheta \cdot \underline{r}_0 - \sin\vartheta \cdot \underline{r}_1 = 0 \qquad (0 < \vartheta < \tfrac{1}{2}\pi).$$

β) $A_0 = 0$. *Punktzykel.* ∞^2. Eine Klasse.

$$\underline{r}_1 = 0.$$

2. $A_1^2 - A_2^2 - A_3^2 = 0$.

α) Nicht alle A_i $(i = 1, 2, 3)$ verschwinden gleichzeitig. *Grenzzykel.* ∞^2. Eine Klasse.

$$\underline{r}_0 + \underline{r}_1 + \underline{r}_2 = 0.$$

β) $A_1 = A_2 = A_3 = 0$. *Absoluter Zykel.* ∞^0.

$$\underline{r}_0 = 0.$$

3. $A_1^2 - A_2^2 - A_3^2 < 0$. *Gleitzykel.* ∞^3. ∞^1 Klassen.

$$\cosh\eta \cdot \underline{r}_0 - \sinh\eta \cdot \underline{r}_3 = 0 \qquad (0 \neq \eta \neq \infty).$$

b. Uneigentliche Zykeln. $(AA) < 0$.

1. $A_0 \neq 0$. *Uneigentlicher Zykel* allgemeiner Art. ∞^3. ∞^1 Klassen.

$$\sinh\eta \cdot \underline{r}_0 - \cosh\eta \cdot \underline{r}_3 = 0 \qquad (0 \neq \eta \neq \infty).$$

2. $A_0 = 0$. *Geradenzykel.* ∞^2. Eine Klasse.

$$\underline{r}_3 = 0.$$

II. *Reduzible Zykeln.* $(AA) = 0$.

1. $A_0 \neq 0$. Mittelpfeil eigentlich. ∞^2. Eine Klasse.

$$\underline{r}_0 + \underline{r}_3 = 0, \qquad \lambda_1\mu_2 = 0.$$

2. $A_0 = 0$. Der Mittelpfeil ist Punktpfeil. ∞^1. Eine Klasse.

$$\underline{r}_1 + \underline{r}_2 = 0, \qquad \lambda_1\mu_1 = 0.$$

Radius des Drehzykels, Grenzabstand des Grenzykels * und Abstand des

* Über die Invarianz dieses Ausdrucks gelten entsprechende Bemerkungen, wie in der vorigen Fussnote. Gegenüber den hyperbolischen Bewegungen, die nur einen einzigen unendlich fernen Punkt y in Ruhe lassen, hat ein zugänglicher Punkt x die Invariante $\{x, y\}$:

$$e^{\{x,y\}} = \frac{x_1 y_1 - x_2 y_2 - x_3 y_3}{\sqrt{x_1^2 - x_2^2 - x_3^2}}.$$

Gleitzykels (über die Gestalten dieser Gebilde vgl. § 3) hängen mit den Grössen 2ϕ der zugehörigen Bewegungen (Drehwinkel, Gleitöffnung, Gleitstrecke) zusammen durch die Gleichung

$$|r| = \log \epsilon \frac{e^{\frac{\epsilon\phi}{2}} + e^{-\frac{\epsilon\phi}{2}}}{e^{\frac{\epsilon\phi}{2}} - e^{-\frac{\epsilon\phi}{2}}},$$

wenn man der Reihe nach setzt:

$$\epsilon^2 = -1, \qquad \epsilon^2 = 0, \qquad \epsilon^2 = +1.$$

Im Falle $\epsilon^2 = -1$ gibt die Gleichung den bekannten Zusammenhang zwischen Lotlänge und Parallelenwinkel. —

Bezeichnet man die Zykeln, die die absolute Spiegelung gestatten, also die Punktzykeln, Geradenzykeln und den zweiten Typus der reduziblen Zykeln als *spezielle* Zykeln, so gilt der Satz:

Ein Zykelbüschel enthält einen einzigen speziellen Zykel, oder es besteht ganz aus solchen.

Im ersten Falle hat der spezielle Zykel einfache geometrische Bedeutung (Mittelpunkt des Drehzykels, gemeinsame Normale der beiden Pfeile, in denen sich zwei Zykel durchdringen, Scheitelpunkt des Grenzzykels u. s. w.).

Jeder zum absoluten Zykel orthogonale Zykel ist speziell.

§ 8. *Konstruktionen im Pfeilgebiet.*

Von der elementaren Pfeilgeometrie kehren wir noch einmal zu allgemeineren Betrachtungen zurück. Es ist zunächst zu zeigen, wie man beliebig viele Pfeile eines durch drei Pfeile bestimmten irreduziblen Zykels konstruieren kann. Daran schliessen wir noch einige andere Konstruktionsaufgaben an.

Irgend drei Pfeile, von denen keine zwei parallel sind, bestimmen, wie schon gesagt, eine Abbildung des absoluten Kegelschnitts auf sich selbst von folgender Eigenschaft:

Das Doppelverhältnis von irgend vier ursprünglichen Punkten ist gleich dem Doppelverhältnis der transformierten Punkte.

Deuten wir jeden ursprünglichen Punkt als Anfangspunkt, den transformierten als Endpunkt eines Pfeils, so erfüllen die ∞^1 auf diese Weise erhaltenen Pfeile den ganzen durch die drei gegebenen Pfeile bestimmten Zykel.

Ist also der Zykel pfeilweise bekannt, so ist dadurch die ganze Abbildung gegeben.

Man kann sich nun auf den Standpunkt stellen, dass nur die *speziellen* Zykeln

Es genügt, dieser Grösse, die wir als *Grenzabstand* des Punktes x gegen den Punkt y bezeichnen, positive Werte beizulegen. Der Ort der Punkte x, die gegen einen unendlich fernen Punkt gleichen Grenzabstand besitzen, ist eine *Grenzkurve*. Der Abstand zweier Grenzkurven, die zu ein und demselben unendlich fernen Punkte gehören, ist gleich der Differenz der Grenzabstände.

unmittelbar gegeben seien, und dann die Aufgabe stellen, *unter Benutzung nur von speziellen Zykeln* die Abbildung herzustellen und damit auch nicht-spezielle Zykeln zu konstruieren.

Das kann auf folgende Weise geschehen: (wir setzen voraus, dass der zu konstruierende Zykel nicht schon speziell ist).

1. Die Anfangspunkte (a, b, c) der drei Pfeile verbinden wir mit einem (nicht unendlich fernen, sonst aber) beliebigen Punkt P_1. Es entsteht ein spezieller Zykel, gegeben durch die drei Pfeile $\overrightarrow{aA}$ $\overrightarrow{bB}$, $\overrightarrow{cC}$.

2. Jetzt verbinden wir einen zweiten (nicht unendlich fernen) Punkt P_2 mit den Punkten A, B und C durch Pfeile. Der neue spezielle Zykel ist durch die Pfeile $\overrightarrow{A'A}$, $\overrightarrow{B'B}$, $\overrightarrow{C'C}$ gegeben.

3. Sind die Punkte P_1 und P_2 passend gewählt, so ist es stets möglich, einen dritten Punkt P_3 so zu finden, dass die orientierten Verbindungsgeraden $\overrightarrow{A'P_3}$, $\overrightarrow{B'P_3}$, $\overrightarrow{C'P_3}$ der Reihe nach durch die Endpunkte a', b', c' der drei gegebenen Pfeile laufen.

Wendet man diese Konstruktion auf einen vierten unendlich fernen Punkt d an, so erhält man den zugeordneten Punkt d' und somit einen vierten Pfeil des durch die drei gegebenen laufenden Zykels.

Die drei Punkte P_1, P_2, P_3 kann man z. B folgendermassen wählen: (Fig. 7)

$$P_1 = (ab, cc'), \qquad P_3 = (aa', bb'), \qquad P_2 = (ab, P_3c').$$

Die Punkte P existieren stets. Vorkommen kann es aber, dass der eine oder andre von ihnen unendlich fern liegt. Der Fall kann dann durch Einschieben noch eines vierten speziellen Zykels stets vermieden werden. Auf gleiche Weise beseitigt man unbequeme Lageverhältnisse.

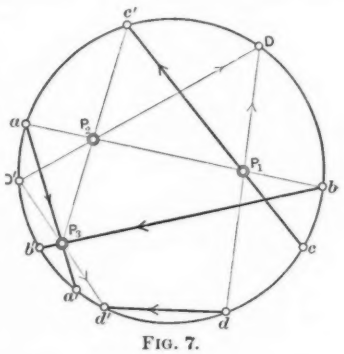


FIG. 7.

Benutzt sind nur die drei *linearen* Fundamentalkonstruktionen der ebenen *projektiven* Geometrie:

1) Zwei Punkte zu verbinden.

2) Zwei Gerade zum Schnitt zu bringen.

3) Den zweiten Schnittpunkt einer Geraden mit einer Kurve zweiter Ordnung zu finden.

Da wir von dem durch die drei Pfeile $\overrightarrow{aa'}$, $\overrightarrow{bb'}$, $\overrightarrow{cc'}$ gehenden Zykel auf Grund dieser linearen Konstruktionen jetzt beliebig viele Pfeile zeichnen können, erklären wir als

1. *Erste (lineare) Fundamentalkonstruktion der Pfeilgeometrie:*
Durch drei Pfeile, von denen keine zwei parallel sind, den Zykel zu legen.

Durch zweimalige Anwendung dieser Konstruktion löst man die Aufgabe:

2. *Einen Pfeil an einem irreduziblen Zykel zu spiegeln.*

Es ist zwar noch ein Pfeil mit einem andern zu *verbinden* (§ 1, Fig. 1); das ist aber keine neue geometrische Konstruktion, da der Pfeil eben als Punktepaar erklärt worden ist; den Träger hatten wir in den Begriff des Pfeils nicht mit aufgenommen.

3. *Es soll die parazyklische (antiparazyklische) Pfeiltransformation konstruiert werden, die drei Pfeile* $\overrightarrow{aA}$, $\overrightarrow{bB}$, $\overrightarrow{cC}$, *von denen keine zwei parallel sind, in die Pfeile* $\overrightarrow{a'A'}$, $\overrightarrow{b'B'}$, $\overrightarrow{c'C'}$ *überführt.*

Man konstruiert im ersten Falle die beiden Zykeln

$$\overrightarrow{aa'}, \overrightarrow{bb'}, \overrightarrow{cc'} \quad \text{und} \quad \overrightarrow{AA'}, \overrightarrow{BB'}, \overrightarrow{CC'};$$

im zweiten Falle dagegen die Zykeln

$$\overrightarrow{aA'}, \overrightarrow{bB'}, \overrightarrow{cC'} \quad \text{und} \quad \overrightarrow{Aa'}, \overrightarrow{Bb'}, \overrightarrow{Cc'}.$$

Das weitere Verfahren, zu einem vierten ursprünglichen Pfeil $\overrightarrow{dD}$ den transformierten zu finden, bedarf keiner Erläuterung mehr.

4. *Den zweiten Durchdringungspfeil zweier irreduzibler Zykeln zu zeichnen, wenn der erste gegeben ist.*

Ein Pfeil $\mathfrak{x}$ wird beliebig gewählt, aber so dass er keinem der beiden gegebenen Zykeln A und B angehört. Dann spiegeln wir ihn an den beiden Zykeln A und B und erhalten die Spiegelbilder $\mathfrak{y}$ und $\mathfrak{z}$. Kurz geschrieben:

$$\mathfrak{x}\{A\}\mathfrak{y}, \qquad \mathfrak{x}\{B\}\mathfrak{z}.$$

Jetzt wird der Zykel $\mathfrak{xyz}$ gezeichnet. Wird der gegebene Durchdringungspfeil I der beiden Zykeln A und B am Zykel $(\mathfrak{xyz})$ gespiegelt, so erhält man den zweiten gesuchten Durchdringungspfeil II:

$$\text{I}\{\mathfrak{xyz}\}\text{ II}.$$

Damit die zuletzt angegebene Spiegelung am Zykel möglich ist, muss natürlich der Zykel $(\mathfrak{xyz})$ irreduzibel sein. So lässt sich aber der Pfeil $\mathfrak{x}$ stets wählen.

5. *Durch einen Pfeil* $\mathfrak{x}$ *den Zykel zu legen, der zu zwei getrennten irreduziblen Zykeln A und B orthogonal ist.*

$$\mathfrak{x}\{A\}\mathfrak{y}, \quad \mathfrak{x}\{B\}\mathfrak{z}.$$

Dann ist der Zykel $(\mathfrak{xyz})$ der gesuchte. Es gibt ∞^1 Lösungen, falls der reduzible Zykel, für den der Pfeil $\mathfrak{x}$ Mittelpfeil ist, mit A und B in einem Zykelbüschel liegt, in allen übrigen Fällen eine einzige Lösung. Das soeben angegebene Verfahren führt jedoch nicht zum Ziel, wenn der Pfeil $\mathfrak{x}$ einem der beiden gegebenen Zykeln angehört. Dann benutzt man am bequemsten Aufgabe 9.

6. *Den Zykel zu zeichnen, der zu zwei getrennten reduziblen Zykeln A und B orthogonal liegt und durch den Pfeil $\underset{\sim}{\mathfrak{x}}$ läuft.*

Man legt durch $\underset{\sim}{\mathfrak{x}}$ und die Mittelpfeile der beiden reduziblen Zykeln den Zykel. Es darf $\underset{\sim}{\mathfrak{x}}$ nicht bereits Mittelpfeil eines der beiden reduziblen Zykeln sein; dann gibt es ∞^1 Lösungen; ebenso wenn die Mittelpfeile der beiden gegebenen reduziblen Zykeln beide zum Pfeil $\underset{\sim}{\mathfrak{x}}$ α-parallel (ω-parallel) sind.

7. *Den Zykel zu zeichnen, der durch den Pfeil $\underset{\sim}{\mathfrak{x}}$ läuft, zu einem irreduziblen und zu einem reduziblen Zykel orthogonal ist.*

Man spiegelt den Pfeil $\underset{\sim}{\mathfrak{x}}$, sowie den Mittelpfeil des reduziblen Zykels am irreduziblen. Dadurch hat man vier Pfeile des gesuchten Zykels. Auch hier kann das Lösungsverfahren versagen (wenn $\underset{\sim}{\mathfrak{x}}$ dem irreduziblen Zykel angehört), ferner gibt es in zwei unschwer zu beschreibenden Fällen ∞^1 Lösungen.

8. *Den Zykel zu zeichnen, der durch einen Pfeil $\underset{\sim}{\mathfrak{x}}$ läuft und dem durch die beiden Zykeln A und B bestimmten Büschel angehört.*

Man benutzt (je nach der Beschaffenheit der beiden Zykeln A und B) eine der drei letzten Aufgaben und zeichnet zwei beliebige Zykeln orthogonal zu A und B. Sie mögen C und D heissen. Sie dürfen nicht reduzibel sein und dürfen nicht durch den gegebenen Pfeil laufen. Weiter:

$$\underset{\sim}{\mathfrak{x}}\{C\}\underset{\sim}{\mathfrak{y}}, \qquad \underset{\sim}{\mathfrak{x}}\{D\}\underset{\sim}{\mathfrak{z}}.$$

Dann ist der Zykel $\underset{\sim}{\mathfrak{x}}\underset{\sim}{\mathfrak{y}}\underset{\sim}{\mathfrak{z}}$ der gesuchte.

9. *Den Orthogonalzykel dreier Zykeln A, B, C, die nicht einem Büschel angehören, zu finden.*

Man wählt einen Pfeil $\underset{\sim}{\mathfrak{x}}$, der keinem der drei Zykeln angehört, beliebig. Durch $\underset{\sim}{\mathfrak{x}}$ legt man die drei Zykeln, die der Reihe nach den Büscheln (B, C), (C, A), (A, B) angehören (vorige Aufgabe). Diese drei Zykeln durchdringen sich ausser in $\underset{\sim}{\mathfrak{x}}$ noch in einem Pfeil $\underset{\sim}{\mathfrak{x}}'$.

Diese Konstruktion wiederholt man mit einem zweiten Pfeil $\underset{\sim}{\mathfrak{y}}$ und erhält den Pfeil $\underset{\sim}{\mathfrak{y}}'$.

Die vier Verbindungspfeile $\underset{\sim}{\mathfrak{x}}\overrightarrow{-}\underset{\sim}{\mathfrak{x}}'$, $\underset{\sim}{\mathfrak{x}}'\overrightarrow{-}\underset{\sim}{\mathfrak{x}}$, $\underset{\sim}{\mathfrak{y}}\overrightarrow{-}\underset{\sim}{\mathfrak{y}}'$, $\underset{\sim}{\mathfrak{y}}'\overrightarrow{-}\underset{\sim}{\mathfrak{y}}$ gehören dem gesuchten Orthogonalzykel an. Durch die soeben angegebene Konstruktion lassen sich auch die oben (bei 5 und 7) erwähnten Fälle erledigen, wenn die dort aufgeführten Methoden versagen.

Als quadratische Konstruktion ist die folgende zu betrachten.

10. *Die beiden Durchdringungspfeile zweier Zykeln zu zeichnen.*

Die beiden Zykeln seien A und B. Man zeichnet zwei weitere Zykeln C und D beliebig, die zu beiden orthogonal sind. Dann bestimmen die drei Zykeln C, D und A einen Orthogonalzykel O. Dieser liegt mit A und B in einem Büschel. Die Durchdringungspfeile von A und B sind daher identisch mit denen von A und O. Die Aufgabe ist also etwas vereinfacht und es gilt als

11. *Zweite (quadratische) Fundamentalkonstruktion der Pfeilgeometrie: Die Durchdringungspfeile zweier orthogonaler Zykeln zu zeichnen.*

Sie ist in dem erklärten Gebiet, in dem es nur reelle Pfeile gibt, natürlich nicht immer lösbar.

§ 9. *Die Hauptarten Nicht-Euklidischer Geometrie der Ebene.*

Um der Pfeilgeometrie eine Stelle im Ganzen der Geometrie zuzuweisen, müssen wir etwas weiter ausholen und z. T. bekannte Dinge ausführen. Dabei werden sich dann gleichzeitig manche der eingeführten Benennungen motivieren.

Die Gesamtheit der eigentlichen reellen Punkte der Ebene kann bekanntlich auf zwei verschiedene Arten zu einem abgeschlossenen Kontinuum ergänzt werden.

A. Zunächst zum *projektiven* Kontinuum. Ein Punkt wird durch *drei* homogene *reelle* Koordinaten $x_1 : x_2 : x_3$ dargestellt. Die zugehörige *achtgliedrige* Transformationsgruppe ist *kontinuierlich:*

$$x_i' = c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3, \qquad (i = 1, 2, 3).$$

B. Das zweite Kontinuum wird dadurch erhalten, dass man das erste doppelt überdeckt:

Zyklisches Kontinuum.* Es hat den Zusammenhang einer singularitätenfreien Fläche zweiter Ordnung, die reelle Punkte, aber keine reellen Erzeugenden besitzt.

Der Punkt des zyklischen Kontinuums kann daher durch *eine komplexe* Zahl $\xi = \xi_1 : \xi_2$ dargestellt werden. Die zugehörige Gruppe ist *sechsgliedrig* (*zyklische* Gruppe = Gruppe der MÖBIUSschen Kreisverwandtschaften). Sie ist *gemischt* und besteht aus zwei getrennten Transformationsscharen:

$$\xi_1' = a_{11}\xi_1 + a_{12}\xi_2, \qquad \xi_1' = b_{11}\bar{\xi}_1 + b_{12}\bar{\xi}_2,$$
$$\xi_2' = a_{21}\xi_1 + a_{22}\xi_2. \qquad \xi_2' = b_{21}\bar{\xi}_1 + b_{22}\bar{\xi}_2.$$

Zyklische Transformationen. *Antizyklische* Transformationen.†

Die zyklische Gruppe ist enthalten in der unendlichen konformen Gruppe:

$$\xi' = f(\xi). \qquad \xi' = f(\bar{\xi}).$$

Konforme Transformationen. *Antikonforme* Transformationen.

A. Man betrachtet nun gewisse Untergruppen der *projektiven* Gruppe, indem man ein absolutes Gebilde (einen irreduziblen *Kegelschnitt*) einführt, und nach den Transformationen fragt, die es in Ruhe lassen. Diese bilden eine *kontinuierliche*‡ dreigliedrige Untergruppe und sollen *Bewegungen* genannt werden.

* Es ist das Kontinuum, mit dem man es in der Theorie der Funktionen einer komplexen Veränderlichen zu tun hat. Man könnte es vielleicht besser als *sphärisch* bezeichnen. Ebenso berechtigt wäre dann aber, wie wir noch sehen werden, der Name *pseudosphärisches* Kontinuum. In unserm Zusammenhang ist die im Text eingeführte Bezeichnung vorzuziehen.

† Durch $\bar{\xi}$ soll, wie üblich, der zu ξ konjugiert-komplexe Wert bezeichnet werden.

‡ Es ist zweckmässig, als *gemischt* nur solche Gruppen zu bezeichnen, deren Erweiterung ins komplexe Gebiet noch gemischt ist.

Der absolute Kegelschnitt besitzt keine reellen Punkte:

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

Elliptische Bewegungen:

$$\dot{x}' = \dot{\alpha} \cdot \dot{x} \cdot \dot{\alpha}.$$

Der absolute Kegelschnitt besitzt reelle Punkte:

$$x_1^2 - x_2^2 - x_3^2 = 0.$$

Hyperbolische Bewegungen:

$$\dot{x}' = \dot{\underline{\alpha}} \cdot \dot{x} \cdot \dot{\alpha}.$$

Dabei bedeutet:

$$\dot{\alpha} = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,$$

$$\dot{\underline{\alpha}} = \alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3,$$

$$\dot{x} = \quad * \quad x_1 e_1 + x_2 e_2 + x_3 e_3.$$

Das benutzte System komplexer Zahlen ist

e_0	e_1	e_2	e_3
e_1	$-e_0$	e_3	$-e_2$
e_2	$-e_3$	$-e_0$	e_1
e_3	e_2	$-e_1$	$-e_0$

Erste Gestalt der Quaternionen (Hamiltonsche Quaternionen).

e_0	e_1	e_2	e_3
e_1	$-e_0$	e_3	$-e_2$
e_2	$-e_3$	e_0	$-e_1$
e_3	e_2	e_1	e_0

Zweite Gestalt der Quaternionen (vgl. § 2).

Die Transformationsdeterminante hat den Wert:

$$(\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^3;$$

Sie ist stets positiv.

$$(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)^3;$$

Sie kann das Vorzeichen wechseln.

Die Gruppe der reellen elliptischen Bewegungen bildet ein einziges Kontinuum.

Die Gruppe der reellen hyperbolischen Bewegungen umfasst zwei getrennte Kontinua:

1) $\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 > 0$. *Eigentliche* Bewegungen.

2) $\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 < 0$. *Uneigentliche* Bewegungen.*

B. Entsprechende Überlegungen stellt man für das *zyklische* Kontinuum an. Ein irreduzibler *Kreis* wird als absolutes Gebilde erklärt.† Er bleibt in Ruhe bei einer dreigliedrigen *gemischten* Gruppe. Diese heisst Gruppe der *Bewegungen* und *Umlegungen*. Die Bewegungen sind die *zyklischen*, die Umlegungen die *antizyklischen* Transformationen der fraglichen Eigenschaft.

* Manche Autoren nennen *Bewegungen* nur die, die wir als eigentliche Bewegungen bezeichnen. Unsere uneigentlichen Bewegungen werden öfter mit den Umlegungen der sphärischen Geometrie in Parallele gesetzt, was aber nicht zutreffend ist.

† Besser gesagt: Als absolutes Polarsystem wird das mit einer gewissen (nicht semidefiniten) binären Hermiteschen Form verbundene erklärt.

2. *Pseudosphärische Geometrie:*

$$x_0 : x_1 : x_2 : x_3 = i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) : \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1.$$

$$x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0.$$

Es folgen jetzt unmittelbar folgende Sätze:

Die Gruppe der *sphärischen* Bewegungen ist *holomorph* zur Gruppe der *elliptischen* Bewegungen.

Die Gruppe der *sphärischen* Bewegungen und Umlegungen ist *meromorph* zur Gruppe der *elliptischen* Bewegungen.

Die Gruppe der *pseudosphärischen* Bewegungen ist *holomorph* zur Gruppe der *hyperbolischen* Bewegungen.

Die Gruppe der *pseudosphärischen* Bewegungen und Umlegungen ist *meromorph* zur Gruppe der *hyperbolischen* Bewegungen.

Der Meromorphismus kommt dadurch zustande, dass der Identität zugeordnet wird die ausgezeichnete Transformation

$$x' = \underline{x},$$

die beide Blätter der Ebene vertauscht.

Ein wesentlicher Unterschied zwischen sphärischer und pseudosphärischer Geometrie besteht in Folgendem:

Die vierte Koordinate x_0, durch deren Einführung aus dem Punkte $x_1 : x_2 : x_3$ des projektiven Kontinuums die beiden *übereinander liegenden* Punkte

$$x_0 : x_1 : x_2 : x_3 \qquad \text{und} \qquad x_0 : -x_1 : -x_2 : -x_3$$

des zyklischen Kontinuums hervorgehen, ist im *sphärischen* Falle *stets* reell; in der pseudosphärischen Geometrie indessen nur, solange $x_1^2 - x_2^2 - x_3^2$ nicht negativ ist.

In der pseudosphärischen Geometrie sind als reell (ausser den Punkten des absoluten Kreises) nur zu bezeichnen die Punkte des doppelt überdeckten "zugänglichen" Gebietes.

Während also die *sphärische Ebene* das ganze zyklische Kontinuum erfüllt, besteht die *pseudosphärische Ebene, die denselben Zusammenhang besitzt*, nur aus den Punkten einer doppelt zählenden kreisförmigen Scheibe, deren Ränder zusammenhängen.

§ 10. *Parasphärische Geometrie.*

Wir stellen jetzt die Frage, was für eine Geometrie entstehen würde, wenn man das *unzugängliche* Gebiet der hyperbolischen Ebene doppelt überdeckt, wobei wieder der absolute Kegelschnitt Verzweigungsmannigfaltigkeit sein soll.

Wir definieren eine vierte Koordinate x_0 jetzt durch die Gleichung

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0.$$

Der absolute Kreis besitzt keine reellen Punkte:

$$\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 = 0.$$

Sphärische Bewegungen:

$$\xi_1' = (\alpha_0 + i\alpha_3)\xi_1 + (\alpha_2 + i\alpha_1)\xi_2,$$
$$\xi_2' = (-\alpha_2 + i\alpha_1)\xi_1 + (\alpha_0 - i\alpha_3)\xi_2.$$

Sphärische Umlegungen:

$$\xi_1' = (-\alpha_2 - i\alpha_1)\bar{\xi}_1 + (\alpha_0 + i\alpha_3)\bar{\xi}_2,$$
$$\xi_2' = (-\alpha_0 + i\alpha_3)\bar{\xi}_1 + (-\alpha_2 + i\alpha_1)\bar{\xi}_2.$$

Der absolute Kreis besitzt reelle Punkte:

$$i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) = 0.$$

Pseudosphärische Bewegungen:

$$\xi_1' = (\alpha_0 - \alpha_3)\xi_1 + (\alpha_1 + \alpha_2)\xi_2,$$
$$\xi_2' = (\alpha_2 - \alpha_1)\xi_1 + (\alpha_0 + \alpha_3)\xi_2.$$

Pseudosphärische Umlegungen:

$$\xi_1' = (\alpha_0 - \alpha_3)\bar{\xi}_1 + (\alpha_1 + \alpha_2)\bar{\xi}_2,$$
$$\xi_2' = (\alpha_2 - \alpha_1)\bar{\xi}_1 + (\alpha_0 + \alpha_3)\bar{\xi}_2.$$

Die $\alpha_i (i = 0, 1, 2, 3)$ sind *reell* zu nehmen.

C. Man stellt sich jetzt die Aufgabe, den Punkt ξ des zyklischen Kontinuums durch *vier reelle* homogene Koordinaten $x_0 : x_1 : x_2 : x_3$ so darzustellen, dass der absolute Kreis der sphärischen (pseudosphärischen) Geometrie übergeht in den absoluten Kegelschnitt der elliptischen (hyperbolischen) Geometrie, dass ferner übereinander liegende Punkte beider Blätter der Ebene dargestellt werden durch die Systeme:

$$x_0 : x_1 : x_2 : x_3 \qquad \text{und} \qquad x_0 : -x_1 : -x_2 : -x_3.$$

Zwischen den x besteht dann eine quadratische Relation; das absolute Gebilde erscheint als Verzweigungsmannigfaltigkeit.* Setzt man jetzt:

$$\dot{x} = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_2,$$

so ergeben sich die einfachen Formeln:

Bewegungen:	Umlegungen:
$x' = \underline{\alpha} \cdot \dot{x} \cdot \alpha,$	$\dot{x}' = \underline{\alpha} \cdot \underline{\dot{x}} \cdot \dot{\alpha}.$

Bei Benutzung der ersten (zweiten) Gestalt der Quaternionen erhält man die sphärischen (pseudosphärischen) Bewegungen und Umlegungen.

Die x hängen mit den ξ und unter sich durch folgende Gleichungen zusammen:

1. *Sphärische Geometrie:*

$$x_0 : x_1 : x_2 : x_3 = \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 : \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1 : i(\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1) : \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2.$$

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0.$$

* Diesen Sachverhalt verkennt Hr. F. KÖLMEL: *Bewegungen und Umlegungen der Ebene bei projektiver Massbestimmung*, Lahr bei Schauenburg, 1900, pag. 18. Wir weisen darauf hin, da Hr. KÖLMEL seine Behauptung, der absolute Kegelschnitt sei ebenfalls doppelt überdeckt, halten zu können glaubt. Auch in einer Strassburger Dissertation [A. MAGENER, *Anallagmatische Punktkoordinaten im Kugelgebüsch und ihre Anwendung auf die Nicht-Euklidische Geometrie*, 1906] werden tatsächlich bestehende Unterschiede verwischt. Der Verfasser hält es für unzulässig, die Punkte der von ihm zusammengefassten Punktepaare einzeln zu betrachten (!)

Ein Punkt $x_0 : x_1 : x_2 : x_3$ dieses Kontinuums (der *parasphärischen Ebene*, wie wir sagen wollen) kann dann durch *zwei reelle* Parameter (λ, μ) in folgender Weise dargestellt werden.

$$-x_0 : x_1 : x_2 : x_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : \lambda_2\mu_2 + \lambda_1\mu_1 : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1.$$

Die "parasphärischen Bewegungen und Umlegungen" werden, wenn man sich der Koordinaten x bedient, durch genau dieselben Gleichungen geliefert, wie die pseudosphärischen:

$$\dot{x}' = \dot{\alpha}\cdot\dot{x}\cdot\dot{\alpha}, \qquad \dot{x}' = \dot{\alpha}\cdot\dot{\underline{x}}\cdot\dot{\alpha}.$$

In den Parametern (λ, μ) lauten sie:

$$\lambda_1' = (\alpha_0 + \alpha_3)\lambda_1 + (\alpha_2 - \alpha_1)\lambda_2, \qquad \mu_1' = (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2,$$
$$\lambda_2' = (\alpha_2 + \alpha_1)\lambda_1 + (\alpha_0 - \alpha_3)\lambda_2, \qquad \mu_2' = (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2.$$

(Bewegungen)

$$\lambda_1' = (\alpha_0 + \alpha_3)\mu_1 + (\alpha_2 - \alpha_1)\mu_2, \qquad \mu_1' = (\alpha_0 + \alpha_3)\lambda_1 + (\alpha_2 - \alpha_1)\lambda_2,$$
$$\lambda_2' = (\alpha_2 + \alpha_1)\mu_1 + (\alpha_0 - \alpha_3)\mu_2, \qquad \mu_2' = (\alpha_2 + \alpha_1)\lambda_1 + (\alpha_0 - \alpha_3)\lambda_2.$$

(Umlegungen)

Die Übereinstimmung ist aber nicht nur formal; die parasphärischen Bewegungen und Umlegungen unterscheiden sich von den pseudosphärischen vielmehr nur durch die Objekte, nämlich durch die verschiedenen Ungleichungen, denen die zu transformierenden Veränderlichen zu genügen haben, während die Transformationsformeln dieselben sind.

Diese Übereinstimmung erstreckt sich aber nicht mehr auf umfassendere Gruppen. Die parasphärischen Bewegungen und Umlegungen lassen sich nämlich zu einer sechsgliedrigen Gruppe erweitern, (*parazyklische Gruppe*) die von der MOEBIUSschen, von uns *zyklisch* genannten Gruppe ganz verschieden ist.

Die *zyklische* Gruppe ist nämlich holomorph zu folgender Gruppe:

$$\dot{x}' = \dot{\alpha}\cdot\dot{x}\cdot\dot{\beta}, \qquad \dot{x}' = \dot{\alpha}\cdot\dot{\underline{x}}\cdot\dot{\beta}.$$

Zyklische Transformationen. *Antizyklische* Transformationen.

Dabei hat man $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ zu setzen und sich der *ersten* Gestalt der Quaternionen zu bedienen.*

Wählen wir nun wieder die zweite Gestalt der Quaternionen, setzen jetzt aber

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0,$$

* Deutet man die x in den letzten Gleichungen als homogene Punktkoordinaten im R_3, so stellen diese Gleichungen die Gruppe der elliptischen Bewegungen und Umlegungen dar. Doch ist der Zusammenhang dieser Gruppe mit den sphärischen (pseudosphärischen) Bewegungen und Umlegungen viel komplizierter als bei den entsprechenden Verhältnissen der parazyklischen Gruppe, wie wir hier nicht näher ausführen können.

so liefern die Gleichungen die *parazyklische* Gruppe, die in den (λ, μ) durch die Gleichungen (9) und (20) ausgedrückt wird. Die beiden Transformationsscharen der Gruppe werden konsequent folgendermassen benannt:

$$\dot{x}' = \dot{\alpha} \cdot \dot{x} \cdot \dot{\beta},$$

Parazyklische Punkttransformationen.

$$\dot{x}' = \dot{\alpha} \cdot \dot{\underline{x}} \cdot \dot{\beta}.$$

Antiparazyklische Punkttransformationen.

Dem Aufbau der zur parazyklischen Gruppe gehörigen Geometrie stehen Schwierigkeiten nicht entgegen. Man wird es aber als Mangel empfinden, dass sie sich im unzugänglichen Gebiet abspielt. Diesen Mangel beseitigt die absolute Korrelation, indem sie aus der Gesamtheit der doppelt überdeckten *unzugänglichen Punkte* hervorgehen lässt die Gesamtheit der doppelt überdeckten *zugänglichen Geraden*. Die doppelte Überdeckung des Geradengebietes vollzieht sich nun vermöge des *Orientierungs*prozesses; die Gerade wird durch zwei eigentliche Pfeile (§ 1) ersetzt. Die Pfeilkoordinaten (§ 1) ergeben sich einfach:

$$\mathfrak{x}_0 : \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3 = -x_0 : -x_1 : x_2 : x_3.$$

Die Pfeilgeometrie erweist sich somit als Bild einer Punktgeometrie, die an und für sich wenig Interesse zu bieten schien, andererseits aber, obwohl sie nicht die reelle Euklidische Geometrie zur Grenzlage hat, doch eine Lücke der vorhandenen Systeme Nicht-Euklidischer Geometrie ausfüllt.—

Unsere Pfeilgeometrie ist ferner eine von drei zusammengehörigen Disziplinen, die als *elliptische*, *hyperbolische* und *Euklidische Pfeilgeometrie* unterschieden werden können. Die hier allein behandelte würde als *hyperbolische* zu bezeichnen sein. Zur elliptischen Pfeilgeometrie gelangt man durch folgende Überlegungen:

Man kann auf der Kugel eine gegenseitig-eindeutige Zuordnung vornehmen zwischen den Punkten und den orientierten Hauptkreisen. Irgend ein Punkt liegt zu dem ihm entsprechenden orientierten Hauptkreis so, wie der Nordpol der Erdkugel zu dem von West nach Ost durchlaufenen Aequator. Gehen wir nun durch Zentralprojektion zur projektiv abgeschlossenen und dann doppelt überdeckten Ebene über, so werden die orientierten Hauptkreise zu orientierten Geraden, *Pfeilen*. Ein Pfeil kann dann also durch *einen komplexen* Parameter dargestellt werden. Dieses *elliptische* Pfeilkontinuum besitzt aber ganz anderen Zusammenhang wie das hyperbolische. Es gibt keine Verzweigungsmannigfaltigkeit. Ein Analogon zu den Punktpfeilen gibt es (im reellen Gebiet) nicht.

Die Gruppe der zyklischen Punkttransformationen geht über in eine Gruppe von zyklischen *Pfeil*transformationen. Die Formeln sind eben nur anders zu deuten.

Als Bewegungen und Umlegungen sind zu erklären diejenigen zyklischen bezw. antizyklischen Pfeiltransformationen, die entgegengesetzte Pfeile wieder in ebensolche überführen.

An die Stelle der bilinearen Formen in Pfeilparametern, deren Nullstellen in der hyperbolischen Pfeilgeometrie die Zykeln lieferten, treten binäre Hermitesche Formen. Den definiten Formen entsprechen *nullteilige* Zykeln, d. i. solche, die keine reellen Pfeile besitzen. Von den reduziblen Zykeln ist nur der Mittelpfeil übrig geblieben.

Verloren geht die Umkehrbarkeit der Beziehung zwischen Bewegungen und irreduziblen Zykeln. Ebenso verschwindet der Begriff: Parallelismus von Pfeilen und die daraus folgenden Tatsachen, z. B. auch die einfache Konstruktion der Spiegelung am Zykel.

Man darf daher wohl die *hyperbolische* Pfeilgeometrie als die interessantere erklären.

Beide Pfeilgeometrieen lassen sich nun ins komplexe Gebiet fortsetzen. Diese ihre Erweiterungen lassen sich dann geradezu identifizieren, wie wir noch kurz andeuten wollen.

Hyperbolische Geometrie:

Absoluter Kegelschnitt reell *mit reellen Punkten.*

$$x_1^2 - x_2^2 - x_3^2 = 0,$$

$$x_1 : x_2 : x_3 = \lambda_1^2 + \lambda_2^2 : \lambda_2^2 - \lambda_1^2 : 2\lambda_1\lambda_2,$$

Elliptische Geometrie:

Absoluter Kegelschnitt reell *ohne reelle Punkte.*

$$x_1^2 + x_2^2 + x_3^2 = 0,$$

$$x_1 : x_2 : x_3 = \lambda_1^2 - \lambda_2^2 : i(\lambda_1^2 + \lambda_2^2) : 2\lambda_1\lambda_2,$$

wobei der Parameter $\lambda_1 : \lambda_2$ jetzt alle möglichen *komplexen* Werte durchläuft. Gehört der Punkt y des absoluten Kegelschnitts zum Parameterwert $\mu_1 : \mu_2$, so kann der *komplexe* Pfeil mit dem Anfangspunkt x und dem Endpunkt y dargestellt werden durch die Koordinaten:

H. G. $$\underset{\sim}{r}_0 : \underset{\sim}{r}_1 : \underset{\sim}{r}_2 : \underset{\sim}{r}_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : -(\lambda_2\mu_2 + \lambda_1\mu_1) : \lambda_2\mu_2 - \lambda_1\mu_1 : \lambda_1\mu_2 + \lambda_2\mu_1,$$

$$\underset{\sim}{r}_0^2 + \underset{\sim}{r}_1^2 - \underset{\sim}{r}_2^2 - \underset{\sim}{r}_3^2 = 0.$$

E. G. $$\underset{\sim}{r}_0 : \underset{\sim}{r}_1 : \underset{\sim}{r}_2 : \underset{\sim}{r}_3 = \lambda_1\mu_2 - \lambda_2\mu_1 : \lambda_1\mu_1 - \lambda_2\mu_2 : i(\lambda_1\mu_1 + \lambda_2\mu_2) : \lambda_1\mu_2 + \lambda_2\mu_1,$$

$$\underset{\sim}{r}_0^2 - \underset{\sim}{r}_1^2 - \underset{\sim}{r}_2^2 - \underset{\sim}{r}_3^2 = 0.$$

Die beiden Kontinua komplexer Pfeile lassen sich nun durchaus eindeutig-umkehrbar aufeinander abbilden, am bequemsten dadurch, dass man solche Pfeile einander zuordnet, die zu gleichen Parameterpaaren $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$ gehören. Wir erklären nun: Ein Pfeil heisst *reell*, wenn seine *Koordinaten* reell sind, d. i.

wenn seine Parameter *einzeln reell* sind:

$$\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1 = 0, \quad \mu_1\bar{\mu}_2 - \mu_2\bar{\mu}_1 = 0.$$

Kontinuum der reellen *hyperbolischen* Pfeilgeometrie.

wenn seine Parameter *entgegengesetzt, reziprok* und *konjugiert-komplex* sind:

$$\lambda_1\bar{\mu}_1 + \lambda_2\bar{\mu}_2 = 0.$$

Kontinuum der reellen *elliptischen* Pfeilgeometrie.

Man erhält nun noch ein drittes ebenes Pfeilkontinuum, indem man nur die *eigentlichen* Geraden der Ebene doppelt überdeckt und den so gewonnenen *eigentlichen* Pfeilen die *uneigentliche* (unendlich ferne) *Gerade* als *uneigentlichen Pfeil* hinzufügt. Die zugehörige Geometrie mit *siebengliedriger* Gruppe umfasst die elementare Euklidische. Das neue Pfeilkontinuum darf daher als *Euklidisches* bezeichnet werden. Es hat den Zusammenhang eines einteiligen irreduziblen *Kegels* zweiter Ordnung. Auf weitere Mannigfaltigkeiten, die als zur Euklidischen Geometrie gehörige Pfeilkontinua erklärt werden können, wollen wir hier nicht eingehen. Das hier erwähnte findet man behandelt in der Dissertation von W. BLASCHKE.*

§ 11. *Die aequidistanten Pfeiltransformationen.*

Um den Zusammenhang zwischen der MOEBIUSschen Geometrie und der parazyklischen bis zu einem gewissen Grade zu erschöpfen, werfen wir noch einen kurzen Blick auf die Differentialgeometrie im Pfeilgebiet der hyperbolischen Ebene und zeigen das Vorhandensein einer *unendlichen* Gruppe von Pfeiltransformationen, die dieselbe Rolle spielt, wie in der MOEBIUSschen Geometrie die *konforme* Gruppe.

Wir führen *inhomogene* Pfeilparameter ein:

$$\lambda = \frac{\lambda_1}{\lambda_2}, \qquad \mu = \frac{\mu_1}{\mu_2}.$$

Dadurch erleiden die folgenden Entwicklungen Einschränkungen, mit deren Angabe wir uns aber nicht aufhalten wollen.

Ein Pfeil (λ, μ) *schneidet* einen benachbarten $(\lambda + d\lambda, \mu + d\mu)$, wenn $d\mu : d\lambda$ positiv ist. Zwei dem Pfeil (λ, μ) benachbarte Pfeile $(\lambda + d_1\lambda, \mu + d_1\mu)$ und $(\lambda + d_2\lambda, \mu + d_2\mu)$ bestimmen eine Strecke η:

$$\operatorname{tgh} \eta = \frac{d_1\lambda \cdot d_2\mu - d_2\lambda \cdot d_1\mu}{d_1\lambda \cdot d_2\mu + d_2\lambda \cdot d_1\mu}.$$

Ist $d_1\mu : d_1\lambda > 0$, $d_2\mu : d_2\lambda > 0$, so ist η die Länge der Strecke, die auf dem ursprünglichen Pfeil von den beiden benachbarten abgeschnitten wird.

Für $d_1\mu : d_1\lambda < 0$, $d_2\mu : d_2\lambda < 0$ bestimmt jeder benachbarte Pfeil mit dem ursprünglichen eine gemeinsame Normale; η ist dann der Abstand dieser beiden Geraden.

Ist endlich $d_1\mu : d_1\lambda > 0$, $d_2\mu : d_2\lambda < 0$, und bezeichnet η' den Abstand des Schnittpunktes $(\lambda, \mu; \lambda_1 + d_1\lambda, \mu + d_1\mu)$ von der gemeinsamen Normalen $(\lambda, \mu; \lambda + d_2\lambda, \mu + d_2\mu)$, so ist

$$\frac{d_1\lambda \cdot d_2\mu - d_2\lambda \cdot d_1\mu}{d_1\lambda \cdot d_2\mu + d_2\lambda \cdot d_1\mu} = \coth \eta'.$$

* W. BLASCHKE, *Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene*; Monatshefte für Mathematik und Physik, 21. Jahrgang (1910). Herr BLASCHKE war so liebenswürdig, zu unserm Aufsatze die Figuren zu zeichnen.

Man erhält nun sofort den interessanten Satz:

Jede Pfeiltransformation

$$\lambda' = F(\lambda), \qquad \mu' = \phi(\mu), \quad (\textit{parakonforme} \text{ Transformation})$$

die α-parallele (ω-parallele) Pfeile in ebensolche überführt, ebenso jede Pfeiltransformation

$$\lambda' = F(\mu), \qquad \mu' = \phi(\lambda), \quad (\textit{antiparakonforme} \text{ Transformation})$$

die α-Parallelismus und ω-Parallelismus vertauscht, ist aequidistant.

Dabei bedeuten F und ϕ irgend welche differenzierbaren Funktionen von λ bezw. μ; aber sie müssen umkehrbar sein, d. i. es müssen durch die beiden im letzten Satze angegebenen Gleichungen wirklich nicht-ausgeartete Transformationen geliefert werden. Diese Transformationen lassen also den Abstand zweier Punkte (Punktzykel) invariant.

Man kann die eben erklärte, *parakonforme* Gruppe noch anders darstellen, wobei der Zusammenhang mit der konformen Gruppe evident wird.

Dazu bedienen wir uns eines komplexen Zahlsystems in zwei Haupteinheiten e_λ und e_μ, dessen Multiplikationsregel durch folgende Gleichungen definiert ist:

$$e_\lambda^2 = e_\lambda, \qquad e_\lambda \cdot e_\mu = e_\mu \cdot e_\lambda = 0, \qquad e_\mu^2 = e_\mu.$$

Dann bilden wir aus Pfeilparametern die komplexen Zahlen

$$z = \lambda \cdot e_\lambda + \mu \cdot e_\mu, \qquad \bar{z} = \mu \cdot e_\lambda + \lambda \cdot e_\mu,$$

deren zweite als zur ersten konjugiert-komplex bezeichnet werden darf. Funktionen dieser komplexen Variabeln werden definiert als Summen von Potenzreihen. Dann lässt sich die parakonforme Gruppe einfach schreiben:

$$z' = f(z), \qquad z' = f(\bar{z}).$$

Auch diese Schreibweise wollen wir noch etwas modifizieren. Setzt man nämlich

$$e_\lambda + e_\mu = 1, \qquad 2\xi = \lambda + \mu,$$

$$e_\lambda - e_\mu = \epsilon, \qquad 2\eta = \lambda - \mu,$$

so erhalten wir:

$$\xi' + \epsilon\eta' = f(\xi + \epsilon\eta), \qquad \xi' + \epsilon\eta' = f(\xi - \epsilon\eta),$$

Parakonforme Transformationen. *Antiparakonforme* Transformationen.

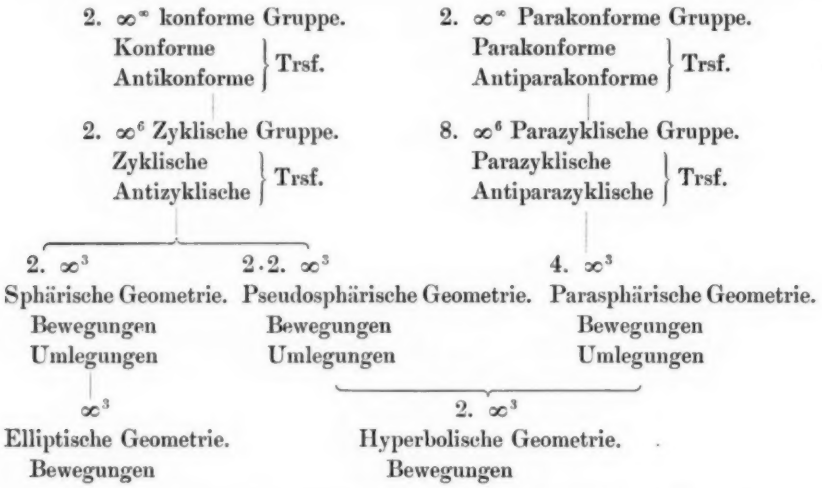
$$\epsilon^2 = +1.$$

Ebenso wie die konforme Gruppe von Punkttransformationen *winkeltreu* ist, ist also die parakonforme Gruppe von Pfeiltransformationen in gewissem Sinne *längentreu*.

In der elliptischen Pfeilgeometrie gedeutet, ist die konforme Gruppe aber ebenfalls längentreu.

Ein korrekter Ausbau dieser hier nur oberflächlich gestreiften Dinge bietet keinerlei Schwierigkeiten. Bemerkt werden muss aber, dass die Benutzung der hier aufgetretenen komplexen Grössen nicht immer vorteilhaft ist.

Wir können jetzt die Pfeilgeometrie in ihrer ganzen Struktur erkennen und stellen die aufgetretenen Gruppen denen der MOEBIUSschen Geometrie gegenüber.



Die aequidistanten Transformationen der Euklidischen Ebene können aus der parakonformen Gruppe durch einen Grenzübergang erhalten werden. Die fragliche unendliche Gruppe ist von Herrn G. SCHEFFERS bestimmt worden; sie führt auf das dritte System komplexer Zahlen in zwei Einheiten, welches von Herrn E. STUDY in der Geometrie der Dynamen verwandt ist.

HANNOVER, 2. *Mai.* 1909.

VECTOR INTERPRETATION OF SYMBOLIC DIFFERENTIAL PARAMETERS*

BY

LOUIS INGOLD

I. Introduction.

The purpose of this paper is to establish a relation between the symbolic theory of invariants of differential forms, due to the late Professor MASCHKE,† and the theory of extensive quantity (vectors), due to GRASSMANN.‡

It will be shown that those symbolic expressions used by MASCHKE which lacked an interpretation in his theory, may be represented as vectors of the GRASSMANN type, and that all of MASCHKE's expressions, including his actual differential parameters, are expressible in the vector system. The theory of such vectors will be extended, new formulas in the symbolic theory will be obtained and applications to geometry will be made.

A preliminary section on space of two dimensions is given, in order that the well known geometric representation of vectors in two dimensions may serve as a basis for a compact treatment of the general theory.

The general theory is stated for three dimensions in concise form, proofs being omitted when their details are simple extensions of the two dimensional case; the obvious generalization to Euclidean space of n dimensions is then indicated.

The less obvious extension to surfaces in three dimensions and, generally, to arbitrary k-dimensional spaces imbedded in an Euclidean space of n dimensions is treated in more detail. This involves an application of the GRASSMANN theory to non-uniform spaces and furnishes an example of a vector system based on variable units.

The paper closes with a section dealing with differential parameters which involve second and higher derivatives of the symbols.

* Presented to the Society (Chicago), March 30, 1907, in somewhat different form, under the title: *Vector theory, in terms of symbolic differential parameters.*

† MASCHKE, *A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 445–469.

‡ H. GRASSMANN, *Ausdehnungslehre*, 1862. References are to GRASSMANN's collected works edited by F. ENGEL.

II. The Two-dimensional Case.

1. *Comparison of the properties of vectors and symbols.* Given any binary quadratic differential form,

$$A \equiv E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2, \tag{1}$$

Maschke replaces the form by a symbolic square

$$A \equiv [f_{(1)}(x, y)dx + f_{(2)}(x, y)dy]^{2*} \tag{2}$$

and writes symbolically

$$f_{(1)}^2(x, y) = E(x, y);\ f_{(1)}(x, y)f_{(2)}(x, y) = F(x, y);\ f_{(2)}^2(x, y) = G(x, y). \tag{3}$$

In the special case $E = G = 1$, $F = 0$, the differential form represents the square of the element of length in a plane; we have then

$$A = ds^2 = dx^2 + dy^2 = (f_{(1)}dx + f_{(2)}dy)^2, \tag{4}$$

where $f_{(1)}$ and $f_{(2)}$ are constant, and where

$$f_{(1)}^2 = 1, \qquad f_{(1)}f_{(2)} = 0, \qquad f_{(2)}^2 = 1. \tag{5}$$

The formulas (5) suggest the properties of so-called *inner* multiplication of a set of unit vectors in a plane. Thus if $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors along the x- and the y-axis respectively, and if $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$ and $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$ are any vectors, the *inner* product is

$$[\boldsymbol{a} \mid \boldsymbol{b}] = a_1b_1 + a_2b_2.\dagger \tag{6}$$

Then for inner multiplication

$$[\boldsymbol{e}_1 \mid \boldsymbol{e}_1] = \boldsymbol{e}_1^2 = 1, \qquad [\boldsymbol{e}_2 \mid \boldsymbol{e}_2] = \boldsymbol{e}_2^2 = 1, \qquad [\boldsymbol{e}_1 \mid \boldsymbol{e}_2] = [\boldsymbol{e}_2 \mid \boldsymbol{e}_1] = 0. \tag{7}$$

On account of this agreement, the *Maschke symbols* $f_{(1)}$ *and* $f_{(2)}$ *may be*

* Maschke, loc. cit., p. 448.

The general quadratic differential form

$$\sum_{i,j=1}^{n} a_{ij}dx_idx_j$$

in n variables $x_1, x_2, \cdots, x_n$ is replaced by $[f_{(1)}dx_1 + f_{(2)}dx_2 + \cdots + f_{(n)}dx_n]^2$, so that symbolically $f_{(i)}f_{(j)} = a_{ij}$; for the special differential form

$$\sum_{i=1}^{n} dx_i^2$$

these equations become

$$f_{(i)}f_{(j)} = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

† The inner product is also defined to be $ab\cos\widehat{\boldsymbol{ab}}$, where a means the length of $\boldsymbol{a}$, i. e., $\sqrt{a_1^2 + a_2^2}$, $b = \sqrt{b_1^2 + b_2^2}$, and $\widehat{\boldsymbol{ab}}$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. This definition is equivalent to the one given in the text.

thought of as a set of unit vectors $\mathbf{e}_1$ *and* $\mathbf{e}_2$, *the multiplications in* (5) *being inner multiplications.*

It will be shown later that this correspondence of properties persists between the Maschke symbols and the vectors of Grassmann's theory in space of n dimensions, every essential property being identical in the two cases.

We proceed just now to show that many — in fact all — of the various symbolic expressions used by Maschke for two dimensions have a meaning in the vector theory, if we set $f_{(1)} = \mathbf{e}_1$ and $f_{(2)} = \mathbf{e}_2$ as suggested above; and that the relations existing between his symbolic expressions correspond to geometric relations.

2. *Notations.* It is found convenient in the symbolic theory to introduce other symbols, $\phi_{(1)}$, $\phi_{(2)}$ equivalent to $f_{(1)}, f_{(2)}$, in the sense that

$$\phi_{(1)}^2 = E, \qquad \phi_{(1)}\phi_{(2)} = F, \qquad \phi_{(2)}^2 = G,$$

according to equations (3). With the aid of two sets of symbols, $f_{(1)}, f_{(2)}$; $\phi_{(1)}, \phi_{(2)}$, expressions quadratic in the E, F, G may be represented symbolically; thus,

$$E^2 = f_{(1)}^2\phi_{(1)}^2, \qquad EF = f_{(1)}^2\phi_{(1)}\phi_{(2)}, \quad 2(EG - F^2) = (f_{(1)}\phi_{(2)} - f_{(2)}\phi_{(1)})^2, \quad \text{etc.}$$

When convenient, still other symbols are introduced, equivalent to f and ϕ, in the sense described above. It is agreed, however, that no more than two symbols of the same kind shall occur in the same product, and that in every product two symbols of like kind, i. e., two f's or two ϕ's, etc., occurring as factors are to be interpreted according to equations (3) [or equations (5) in the special case we are now considering]. Otherwise the symbols are subject to the same laws as ordinary algebraic quantities. The discriminant $EG - F^2$ of the general quadratic form (1) is denoted by $1/\beta^2$.* In general, β is a function of x and y; but $\beta = 1$ for the special differential form (4).

The symbols $f_{(1)}, f_{(2)}$ were regarded by Maschke as partial derivatives of a symbolic function $f(x, y)$, so that $f_{(1)} = \partial f/\partial x$, $f_{(2)} = \partial f/\partial y$.†

In the special case (4), since $f_{(1)}$ and $f_{(2)}$ are constant we may write $f = f_{(1)}x + f_{(2)}y$.

The Jacobian

$$\begin{vmatrix} U_{(1)} & V_{(1)} \\ U_{(2)} & V_{(2)} \end{vmatrix}$$

of any two functions U and V is of frequent occurrence and will be denoted briefly by $\{U, V\}$. Still more frequently occurs the expression $\beta\{U, V\}$,

*This notation is also used for the discriminant of the quadratic differential form in n variables.

†In general throughout this paper, subscripts in parentheses will denote partial differentiation. Thus $U(x, y)$ being any function of x and y, $U_{(1)} = \partial U/\partial x$, $U_{(2)} = \partial U/\partial y$. This differs slightly from the notation of MASCHKE, as he used subscripts without parentheses.

β being defined as above; this expression is denoted by (U, V); for our special case, since $\beta = 1$, we have $(U, V) = \{U, V\}$.*

If U and V are invariants of the differential form then (U, V) is again an invariant.†

3. *Differential parameters involving linear functions.* Consider the system of straight lines

$$a(x, y) = a_1 y - a_2 x = \text{const.}$$

parallel to the vector

$$\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2.$$

Since $f_{(1)} = \boldsymbol{e}_1$ and $f_{(2)} = \boldsymbol{e}_2$ we may write $\boldsymbol{a} = (f, a)$.‡

Hence, *if a is any linear function, the differential parameter (f, a) may be interpreted as a vector parallel to the system of straight lines $a = const.$*

In particular, $(f, y) = \boldsymbol{e}_1$, $(f, -x) = \boldsymbol{e}_2$.

Other differential parameters expressed in symbolic form are $(f, a)(f, b)$, $(f, \phi)(f, a)$, $(f, \phi)(f, a)(\phi, b) = (a, b)$, where a and b are functions of x and y, and f and ϕ are symbolic functions of the differential form. We proceed to find their vector meaning when a and b are linear.

The product $(f, a)(f, b)$, when expanded, equals $a_1 b_1 + a_2 b_2$, where as above $a = a_1 y - a_2 x$, and $b = b_1 y - b_2 x$.

Comparing with the definition, p. 450, we see that this result is the *inner* product of the two vectors $\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2$ and $\boldsymbol{b} = b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2$. The product $(\phi, f)(\phi, a)$, when expanded, equals $-f_{(1)} a_2 + f_{(2)} a_1$. Interpreting $f_{(1)}, f_{(2)}$ as before, we see that this is a vector perpendicular to $\boldsymbol{a} = (f, a)$; with Grassmann we call this the *complement* § of the vector $\boldsymbol{a}$ and denote it by $|\boldsymbol{a}$; thus

$$(\phi, f)(f, a) = |\boldsymbol{a}.$$

Finally the expression‖

$$(f, \phi)(f, a)(\phi, b) = (a, b) = a_1 b_2 - a_2 b_1$$

is the so-called *outer* product ¶ of $\boldsymbol{a}$ and $\boldsymbol{b}$.

4. *Differential parameters involving arbitrary functions.* We investigate now the meaning of these same differential parameters when the functions a and b are arbitrary functions of x and y.

The equation of the tangent to the curve $a = \text{const.}$ at a given point x_0, y_0 on

* The same notation is used in the n-dimensional case; $\{a_1, \cdots, a_n\}$ represents the Jacobian of the functions $a_1(x_1, \cdots, x_n), \cdots, a_n(x_1, \cdots, x_n)$, and $(a_1, \cdots, a_n) = \beta\{a_1, \cdots, a_n\}$.

† For definitions of "Invariant of a differential quantic," "differential parameter," etc., see MASCHKE, loc. cit., p. 446.

‡ Notice that $a_{(1)} = \partial a/\partial x = -a_2$, and $a_{(2)} = \partial a/\partial y = a_1$.

§ For the general definition of complement see p. 457.

‖ Cf. MASCHKE, loc. cit., p. 453, equation (34).

¶ The outer product $[\boldsymbol{a}, \boldsymbol{b}]$ of two vectors is usually defined to be $A \cdot B \cdot \sin \widehat{\boldsymbol{ab}}$, where A and B are the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. This is easily seen to be the same as $a_1 b_2 - a_2 b_1$.

the curve is

$$x\left(\frac{\partial a}{\partial x}\right)_{\substack{x=x_0\\y=y_0}} + y\left(\frac{\partial a}{\partial y}\right)_{\substack{x=x_0\\y=y_0}} = k = \text{const.},$$

or

$$x\,[a_{(1)}]_{\substack{x=x_0\\y=y_0}} + y\,[a_{(2)}]_{\substack{x=x_0\\y=y_0}} = k.$$

Let

$$L = x\,[a_{(1)}]_{\substack{x=x_0\\y=y_0}} + y\,[a_{(2)}]_{\substack{x=x_0\\y=y_0}};$$

then

$$(f, L) = (f, a)_{\substack{x=x_0\\y=y_0}} = f_{(1)}[a_{(2)}]_{\substack{x=x_0\\y=y_0}} - f_{(2)}[a_{(1)}]_{\substack{x=x_0\\y=y_0}};$$

but, by Art. 3, (f, L) is directed parallel to the straight lines $L = \text{const.}$, and these straight lines are parallel to the tangent of that curve of the system $a = \text{const.}$ which passes through the point x_0, y_0; hence we have

THEOREM I. *If the symbols $f_{(1)}, f_{(2)}$ of the differential form $dx^2 + dy^2$ are interpreted as unit vectors along the x- and y-axes respectively, then the symbolic differential parameter (f, a) may be interpreted as a vector function of position, tangent at each point of the plane to the curve $a = \text{const.}$ which passes through that point.*

As in the case where a and b were linear, the product $(f, a)(f, b)$ may be regarded as the *inner* product of the two vectors represented by (f, a) and (f, b). The expression $(\phi, f)(\phi, a)$ represents a vector perpendicular to (f, a) and therefore normal to the curve $a = \text{const.}$; and the expression $(f, \phi)(f, a)(\phi, b)$ is the *outer* product of the vectors (f, a), (f, b).

The length* of the vector represented by (f, a) is

$$\sqrt{(f, a)^2} = \sqrt{a_{(1)}^2 + a_{(2)}^2}.$$

The angle θ between two curves, $a = \text{const.}$, $b = \text{const.}$, is given by †

$$\cos\theta = \frac{(f, a)(f, b)}{\sqrt{(f, a)^2}\,\sqrt{(\phi, b)^2}} = \frac{a_{(1)}b_{(1)} + a_{(2)}b_{(2)}}{\sqrt{a_{(1)}^2 + a_{(2)}^2}\,\sqrt{b_{(1)}^2 + b_{(2)}^2}}$$

or by ‡

$$\sin\theta = \frac{(a, b)}{\sqrt{(f, a)^2}\,\sqrt{(\phi, b)^2}} = \frac{a_{(1)}b_{(2)} - a_{(2)}b_{(1)}}{\sqrt{a_{(1)}^2 + a_{(2)}^2}\,\sqrt{b_{(1)}^2 + b_{(2)}^2}}.$$

5. *Change of variables.* We obtain greater generality for the plane by changing to curvilinear coördinates. Let us make the substitution

$$u = u(x, y), \qquad v = v(x, y);$$

we have then

$$du = u_{(1)}dx + u_{(2)}dy, \qquad dv = v_{(1)}dx + v_{(2)}dy.$$

* Cf. p. 450, second footnote.

† Cf. p. 450, second footnote.

‡ Cf. p. 452, last footnote.

Solving for dx and dy, we have

$$dx = \frac{v_{(2)}du - u_{(2)}dv}{(u, v)}, \qquad dy = \frac{u_{(1)}dv - v_{(1)}du}{(u, v)}.$$

Substituting these values in the expression for ds, we have

$$ds = f_{(1)}dx + f_{(2)}dy = \frac{(f, v)}{(u, v)}du + \frac{(u, f)}{(u, v)}dv. \tag{8}$$

By Theorem I, the coefficients of du and dv represent vectors tangent at any point u_0, v_0 to the curves $v = \text{const.}$, $u = \text{const.}$, respectively.

Suppose that the vector function $f(x, y)$ transforms into the vector function $t(u, v)$; we have

$$f_{(1)} = t_{(u)}u_{(1)} + t_{(v)}v_{(1)}, \qquad f_{(2)} = t_{(u)}u_{(2)} + t_{(v)}v_{(2)}.$$

Solving for $t_{(u)}$ and $t_{(v)}$, we have

$$t_{(u)} = \frac{(f, v)}{(u, v)}, \qquad t_{(v)} = \frac{(u, f)}{(u, v)}. \tag{9}$$

The quantic giving length of arc

$$ds^2 = (f_{(1)}dx + f_{(2)}dy)^2 = dx^2 + dy^2$$

transforms into

$$(t_{(u)}du + t_{(v)}dv)^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{10}$$

where $t_{(u)}^2 = E$, $t_{(u)}t_{(v)} = F$, $t_{(v)}^2 = G$, so that the function $t(u, v)$ into which $f(x, y)$ transforms is a symbolic function of the transformed differential quantic; hence

THEOREM II.* *If $A \equiv Edu^2 + 2Fdudv + Gdv^2$ is the differential form giving length of arc in a plane, the parameters u and v being arbitrary, then the derivatives $t_{(u)}$, $t_{(v)}$ of a symbolic function of the form A may be interpreted as vector functions of position, of length $\sqrt{E}$, $\sqrt{G}$, respectively, tangent at each point of the plane to the parameter curves which pass through that point.*

This gives rise to a vector system based on two *units*,

$$\mathbf{e}_1 = t_{(u)}, \qquad \mathbf{e}_2 = t_{(v)},$$

tangent to the parameter curves $v = \text{const.}$, $u = \text{const.}$, respectively; these *units* are themselves variable from point to point, both in length and in direction.

Again, remembering the factor $\beta = (EG - F^2)^{-\frac{1}{2}}$, we have

$$(f, a) = (t, a), \qquad (f, a)(f, b) = (t, a)(t, b),\dagger \quad \text{etc.}, \tag{11}$$

where the expressions on the right involve derivatives with respect to x and

*Cf. Theorem V.

†Since (f, a), (f, b), etc., are invariant expressions. See MASCHKE, loc. cit., p. 449.

y, while the expressions on the left involve derivatives with respect to u and v. These expressions, therefore, have always the same geometric interpretation, even when the parameters u, v are changed. In particular, the expression $(t, \tau)(t, a)$, where t and τ are equivalent symbols of the quantic

$$Edu^2 + 2Fdudv + Gdv^2,$$

still represents the *complement* of the vector (t, a), so that in this notation the operation of taking the complement is invariant.*

III. Vectors in Three Dimensions; Axioms.

6. *Fundamental notions.* In order to extend the preceding work to space of three dimensions, we shall explain briefly a vector system in that space, following in spirit Grassmann.

Let us start with three mutually perpendicular linear vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, which are of unit length and lie along the three axes; equivalence of such linear vectors, their addition and their multiplication follow the rules for two dimensions. The expression $a\boldsymbol{e}_1 + b\boldsymbol{e}_2 + c\boldsymbol{e}_3$, where a, b, c are ordinary numbers, represents a unique vector defined by these rules, namely, the vector from the origin to the point a, b, c, or any parallel vector of equal length; conversely, any vector in space is expressible in that form.

In order to multiply vectors, we may proceed as in ordinary algebra, provided the fundamental products of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in pairs are defined; we shall introduce three of these products

$$\boldsymbol{E}_1 = [\boldsymbol{e}_2, \boldsymbol{e}_3], \qquad \boldsymbol{E}_2 = [\boldsymbol{e}_3, \boldsymbol{e}_1], \qquad \boldsymbol{E}_3 = [\boldsymbol{e}_1, \boldsymbol{e}_2] \tag{12}$$

as new units, and call them *vectors of the second order.* Each may be thought of geometrically as a rectangle of which the factor vectors are sides, or as any equivalent area on any parallel plane. The other products are defined to be

$$[\boldsymbol{e}_3, \boldsymbol{e}_2] = -\boldsymbol{E}_1 = -[\boldsymbol{e}_2, \boldsymbol{e}_3], \qquad [\boldsymbol{e}_1, \boldsymbol{e}_3] = -\boldsymbol{E}_2, \qquad [\boldsymbol{e}_2, \boldsymbol{e}_1] = -\boldsymbol{E}_3. \tag{13}$$

The products used in (12) and (13) are called *outer* products. It is to be noted that the area of the rectangles mentioned is equal to the product of the lengths of the sides into the sine of the included angle,—a rule given in two dimensions on p. 452, footnote. This rule applied to the remaining products gives

$$[\boldsymbol{e}_1, \boldsymbol{e}_1] = [\boldsymbol{e}_2, \boldsymbol{e}_2] = [\boldsymbol{e}_3, \boldsymbol{e}_3] = 0. \tag{14}$$

The outer product of *any* two vectors is obtained in terms of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ by

* The Grassmann definition of *complement* is given in terms of the *units*. In general (i. e., except for special transformations) the complement of a vector referred to a new system of units differs from the complement referred to the original units.

multiplying algebraically the expressions for the two vectors: thus if

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \qquad \boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3,$$

then

$$(15) \quad [\boldsymbol{a}, \boldsymbol{b}]^* = (a_1b_2 - a_2b_1)\boldsymbol{E}_3 + (a_2b_3 - a_3b_2)\boldsymbol{E}_1 + (a_3b_1 - a_1b_3)\boldsymbol{E}_2.$$

It is usual to represent such a sum as occurs in (15) geometrically by a rule for addition of vectors of the *second* order, similar to the familiar parallelogram law for linear vectors; the parallelogram being replaced by a parallelopiped determined by the two addenda, and the sum being represented by a diagonal parallelogram of the parallelopiped; thus the expression $k_1\boldsymbol{E}_1 + k_2\boldsymbol{E}_2 + k_3\boldsymbol{E}_3$ may be represented by a certain parallelogram, or by an equivalent area on any parallel plane. It should be noted that the sign of this vector is reversed by reversing the sense in which the perimeter is directed. It follows that the outer product $[\boldsymbol{a}, \boldsymbol{b}]$ of any two linear vectors may be represented by the parallelogram which they determine, with its perimeter taken in the direction shown by $\boldsymbol{a}$.

The outer product of all three units might be represented in a similar fashion, as a vector of the *third* order; geometrically, a parallelopiped of edges $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$; it is more usual, however, to notice that all such vectors in space of three dimensions are numerical multiples of one another. It is customary, therefore, to characterize such a vector solely by the number which expresses its volume; we define, then,

$$(16) \quad \begin{aligned} &[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = [\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1] = [\boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_2] = 1, \\ &[\boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_1] = [\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_2] = [\boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3] = -1. \end{aligned}$$

The outer product of *any* three linear vectors can be obtained either by algebraic multiplication and use of (16)† or by computing the volume of the parallelopiped determined by the three vectors; thus if

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3, \quad \boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3, \quad \boldsymbol{c} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3,$$

then

$$(17) \quad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and, using parentheses as equivalent to brackets, we also write

$$(18) \quad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = ([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) = (\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]).$$

On occasion, however, we shall distinguish temporarily 3-dimensional vectors from ordinary numbers.

* Cf. definition for two dimensions, p. 452.

† A product in which one of the units is repeated is zero, since the volume of the corresponding parallelopiped is zero.

7. *Complements; regressive and inner products.* Let $\boldsymbol{E}$ be the outer product of several of the unit vectors; we define the complement of $\boldsymbol{E}$, written $|\boldsymbol{E}$, to be the outer product of the unit vectors not used in forming $\boldsymbol{E}$, with a sign such that $[\boldsymbol{E}, |\boldsymbol{E}] = +1$.

It follows* that $|\boldsymbol{E}^{(i)} = [\boldsymbol{E}^{(i)}, \boldsymbol{E}^{(3-i)}]\boldsymbol{E}^{(3-i)}$, where $\boldsymbol{E}^{(3-i)}$ is the outer product of the unit vectors not used in forming $\boldsymbol{E}^{(i)}$, taken in *any* order.

The complement of any linear function of several vectors is defined to be the same linear function of their complements; thus†

$$|(A_1\boldsymbol{E}_1 + A_2\boldsymbol{E}_2) = A_1|\boldsymbol{E}_1 + A_2|\boldsymbol{E}_2. \tag{18}$$

The outer product of two vectors $\boldsymbol{A}, \boldsymbol{B}$ has been defined only for the case in which the sum of the orders of $\boldsymbol{A}$ and $\boldsymbol{B}$ does not exceed three. In case this sum is greater than three, the sum of the orders of $|\boldsymbol{A}$ and $|\boldsymbol{B}$ is less than three; hence in this case $[|\boldsymbol{A}, |\boldsymbol{B}]$ is defined. In terms of this product we define $[\boldsymbol{A}, \boldsymbol{B}]$.

Definition. If the sum of the orders of $\boldsymbol{A}$ and $\boldsymbol{B}$ exceeds three, then the outer product $[\boldsymbol{A}, \boldsymbol{B}]$ is a vector, $\boldsymbol{C}$, such that $|\boldsymbol{C} = [|\boldsymbol{A}, |\boldsymbol{B}]$; in this case the product is called *regressive;* when the sum of the orders of the factors is less than three, the product is called *progressive;* both are called outer products.

Definition. The outer product‡ $[\boldsymbol{A}, |\boldsymbol{B}]$ of a vector $\boldsymbol{A}$ and the complement of another vector, $\boldsymbol{B}$, is called the inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$.§

8. *Vector formulas.* The following formulas enable us to determine the vector represented by a given regressive product. First we consider the regressive product of two units of the second order $([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k])$. We have by definition

$$\begin{aligned} |([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k]) &= ([\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_k, [\boldsymbol{e}_i, \boldsymbol{e}_k, \boldsymbol{e}_j]\boldsymbol{e}_j) \\ &= -[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]^2[\boldsymbol{e}_k, \boldsymbol{e}_j] = [\boldsymbol{e}_j, \boldsymbol{e}_k]. \end{aligned}$$

This by definition is the complement of $[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_i$; hence

$$([\boldsymbol{e}_i, \boldsymbol{e}_j], [\boldsymbol{e}_i, \boldsymbol{e}_k]) = [\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k]\boldsymbol{e}_i.\| \tag{19}$$

Next consider the product $([\boldsymbol{e}_i, \boldsymbol{a}], [\boldsymbol{e}_i, \boldsymbol{e}_k])$ where $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$.

* Capital letters in black face type will be used to denote vectors of any order. If it is desired to specify the order an upper index in parentheses will be employed; thus $\boldsymbol{E}^{(i)}$ means a vector of the i-th order. Small black face letters invariably represent linear vectors.

† From the definition of complement and formula (18) it follows that $|(|\boldsymbol{A}) = \pm\boldsymbol{A}$.

‡ It is customary to omit the comma and regard the complement sign $|$ as the sign of inner multiplication.

§ This definition, and others of this section are in accord with the corresponding definitions for two dimensions in Art. 2. The definitions as stated here are at once extensible to n dimensions.

‖ We have here assumed i, j, k all different, but the formula holds if any two, or if all three are equal, for in that case both sides reduce to zero.

Substituting this expression for $\boldsymbol{a}$ we have

$$(20)\quad \begin{aligned}([\boldsymbol{e}_i, \boldsymbol{a}],\ [\boldsymbol{e}_i, \boldsymbol{e}_k]) &= a_i([\boldsymbol{e}_i, \boldsymbol{e}_i],\ [\boldsymbol{e}_i, \boldsymbol{e}_k]) + a_j([\boldsymbol{e}_i, \boldsymbol{e}_j],\ [\boldsymbol{e}_i, \boldsymbol{e}_k]),\\ &+ a_k([\boldsymbol{e}_i, \boldsymbol{e}_k],\ [\boldsymbol{e}_i, \boldsymbol{e}_k]) = a_j([\boldsymbol{e}_i, \boldsymbol{e}_j],\ [\boldsymbol{e}_k, \boldsymbol{e}_i]) = [\boldsymbol{e}_i, \boldsymbol{a}, \boldsymbol{e}_k]\boldsymbol{e}_i.\end{aligned}$$

Continuing we find, for any three vectors, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$,

$$(21)\qquad ([\boldsymbol{a}, \boldsymbol{b}],\ [\boldsymbol{a}, \boldsymbol{c}]) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a}.$$

Formulas (19) and (20) are special cases of (21).

We are now able to see the geometric meaning of a regressive product $[\boldsymbol{A}, \boldsymbol{B}]$ in case $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors of the second order; let the planes in which $\boldsymbol{A}$ and $\boldsymbol{B}$ are supposed to lie be produced to meet, and let $\boldsymbol{a}$ be a vector in their line of intersection; then, by properly choosing $\boldsymbol{b}$ and $\boldsymbol{c}$, we may write $\boldsymbol{A} = [\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{B} = [\boldsymbol{a}, \boldsymbol{c}]$. By (21), then, we see that $[\boldsymbol{A}, \boldsymbol{B}]$ is a vector of the first order lying in (or parallel to) the line of intersection of the planes of $\boldsymbol{A}$ and $\boldsymbol{B}$.*

The inner product of two vectors

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i\boldsymbol{e}_i, \qquad \boldsymbol{b} = \sum_{i=1}^{3} b_i\boldsymbol{e}_i$$

may be found by applying the definition and multiplying out algebraically; thus

$$(22)\qquad [\boldsymbol{a}\,|\,\boldsymbol{b}] = \left[\sum_{i=1}^{3} a_i\boldsymbol{e}_i,\ \sum_{i=1}^{3} b_i|\boldsymbol{e}_i\right] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]\sum_{i=1}^{3} a_ib_i = a_1b_1 + a_2b_2 + a_3b_3.$$

This is seen to be the extension to three dimensions of the definition of inner product given on p. 450.

Again, the inner product $([\boldsymbol{a}, \boldsymbol{b}]\,|\,\boldsymbol{c})$ may be found similarly to be

$$(a_1b_2 - a_2b_1)(c_1\boldsymbol{e}_2 - c_2\boldsymbol{e}_1) + (a_2b_3 - a_3b_2)(c_2\boldsymbol{e}_3 - c_3\boldsymbol{e}_2) + (a_3b_1 - a_1b_3)(c_3\boldsymbol{e}_1 - c_1\boldsymbol{e}_3).$$

If the inner product of this vector and a vector

$$\boldsymbol{d} = \sum_{i=1}^{3} d_i\boldsymbol{e}_i$$

be formed, the result is

$$\begin{aligned}\{([\boldsymbol{a}, \boldsymbol{b}]\,|\,\boldsymbol{c})\,|\,\boldsymbol{d}\} &= (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)\\ &+ (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2) + (a_3b_1 - a_1b_3)(c_3d_1 - c_1d_3)\end{aligned}$$

and this same result is found for the inner product $([\boldsymbol{a}, \boldsymbol{b}]\,|\,[\boldsymbol{c}, \boldsymbol{d}])$; hence

$$(23)\qquad ([\boldsymbol{a}, \boldsymbol{b}]\,|\,[\boldsymbol{c}, \boldsymbol{d}]) = \{([\boldsymbol{a}, \boldsymbol{b}]\,|\,\boldsymbol{c})\,|\,\boldsymbol{d}\}.$$

We proceed to obtain certain other fundamental formulas: let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be any three independent linear vectors, and $\boldsymbol{d}$ any other linear vector. Then we may

* In n dimensions if the product $[\boldsymbol{A}, \boldsymbol{B}]$ is progressive, it represents a vector in the least space containing $\boldsymbol{A}$ and $\boldsymbol{B}$; if regressive, a vector in the greatest space common to $\boldsymbol{A}$ and $\boldsymbol{B}$.

write

$$\boldsymbol{d} = l\boldsymbol{a} + m\boldsymbol{b} + n\boldsymbol{c},$$

where l, m, n are ordinary numbers. To obtain l, m, n, take the outer product of $\boldsymbol{d}$ with $[\boldsymbol{a}, \boldsymbol{b}]$, $[\boldsymbol{b}, \boldsymbol{c}]$, $[\boldsymbol{c}, \boldsymbol{a}]$, respectively. We thus obtain

$$(24)\quad [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}] = n[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}],\quad [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}] = l[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}],\quad [\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d}] = m[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}];$$

hence

$$(25)\qquad l = \frac{[\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]},\qquad m = \frac{[\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]},\qquad n = \frac{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.$$

Again, taking the outer product of $\boldsymbol{d}$ with $\boldsymbol{c}$, we have

$$[\boldsymbol{c}, \boldsymbol{d}] = l[\boldsymbol{c}, \boldsymbol{a}] - m[\boldsymbol{b}, \boldsymbol{c}];$$

hence

$$(26)\qquad \begin{aligned}([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) &= -l([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{a}, \boldsymbol{c}]) + m([\boldsymbol{b}, \boldsymbol{a}], [\boldsymbol{b}, \boldsymbol{c}]) \\ &= -l[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - m[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{b}^* = [\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]\boldsymbol{a}.\end{aligned}$$

Similarly

$$[\boldsymbol{a}, \boldsymbol{b}] = \frac{l}{m}[\boldsymbol{a}, \boldsymbol{d}] - \frac{n}{m}[\boldsymbol{a}, \boldsymbol{c}];$$

hence

$$(27)\qquad ([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}]) = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{d}.$$

If we represent the two-dimensional vector $[\boldsymbol{c}, \boldsymbol{d}]$ in (26) as the complement of a linear vector $\boldsymbol{v}$, we have

$$(28)\qquad ([\boldsymbol{a}, \boldsymbol{b}]\,|\boldsymbol{v}) = [\boldsymbol{a}\,|\boldsymbol{v}]\boldsymbol{b} - [\boldsymbol{b}\,|\boldsymbol{v}]\boldsymbol{a}.$$

Taking the inner product of both sides of this equation by another vector $\boldsymbol{w}$, and using formula (23), we have

$$(29)\quad ([\boldsymbol{a}, \boldsymbol{b}]\,|[\boldsymbol{v}, \boldsymbol{w}]) = [\boldsymbol{a}|\boldsymbol{v}][\boldsymbol{b}|\boldsymbol{w}] - [\boldsymbol{b}|\boldsymbol{v}][\boldsymbol{a}|\boldsymbol{w}] = \begin{vmatrix} [\boldsymbol{a}|\boldsymbol{v}], [\boldsymbol{a}|\boldsymbol{w}] \\ [\boldsymbol{b}|\boldsymbol{v}], [\boldsymbol{b}|\boldsymbol{w}] \end{vmatrix}.$$

The vector formulas just given for three dimensions are readily extended to n dimensions. The extensions of formulas (27), (28), and (29) are written down here for reference. They may be proved by methods analogous to those just indicated for three dimensions.

Let $[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k]$ and $[\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r]$ be two vectors in a space of n dimensions, of orders k and r respectively, and suppose $k + r > n$, so that the product $([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r])$ is regressive. The extension of (27) may then be written

$$(27_n)\qquad \begin{aligned}&([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r]) \\ &\qquad = \sum_c [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k, \boldsymbol{b}_{c_1}, \cdots, \boldsymbol{b}_{c_{n-k}}][\boldsymbol{b}_{c_{n-k+1}}, \cdots, \boldsymbol{b}_{c_r}]\dagger\end{aligned}$$

* Cf. formula (21), p. 458.

† *Ausdehnungslehre*, p. 83, no. 113.

where $c_1, \cdots, c_{n-k}$ stands for a combination of $n-k$ of the subscripts $1, 2, \cdots, r$, and $c_1, c_2, \cdots, c_r$ are taken in such order that $[\boldsymbol{b}_{c_1}, \boldsymbol{b}_{c_2}, \cdots, \boldsymbol{b}_{c_r}] = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r]$. We have also

$$(28_n) \qquad ([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{k+r}] \mid [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_r]) = \sum_c ([\boldsymbol{a}_{c_1}, \cdots, \boldsymbol{a}_{c_r}] \mid [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r])[\boldsymbol{a}_{c_{r+1}}, \cdots, \boldsymbol{a}_{c_{r+k}}],^*$$

$$(29_n) \qquad ([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k] \mid [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k]) = \begin{vmatrix} [\boldsymbol{a}_1 \mid \boldsymbol{b}_1], & [\boldsymbol{a}_1 \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_1 \mid \boldsymbol{b}_k] \\ [\boldsymbol{a}_2 \mid \boldsymbol{b}_1], & [\boldsymbol{a}_2 \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_2 \mid \boldsymbol{b}_k] \\ \vdots & \vdots \\ [\boldsymbol{a}_k \mid \boldsymbol{b}_1], & [\boldsymbol{a}_k \mid \boldsymbol{b}_2] \cdots [\boldsymbol{a}_k \mid \boldsymbol{b}_k] \end{vmatrix}.^\dagger$$

9. *Axioms.* The essential properties‡ used in what precedes may be stated in the following axioms, which will be seen to ensure the preceding facts for three dimensions, and which also furnish a sufficient basis for an immediate extension of all these ideas to space of n dimensions for a system of n mutually orthogonal linear unit vectors. The notation explained in footnote, p. 457, will be employed. Small letters in ordinary type denote ordinary numbers.

I. If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are vectors of the r-th order, $r \leqq n$, then

(1) $l\boldsymbol{A} = \boldsymbol{A}l$;
(2) $l(m\boldsymbol{A}) = lm\boldsymbol{A}$;
(3) $0 \cdot \boldsymbol{A} = 0$;
(4) $l\boldsymbol{A} = 0$ implies $l = 0$ or $\boldsymbol{A} = 0$;
(5) $l\boldsymbol{A} + m\boldsymbol{B}$ is a uniquely determined vector of order r;
(6) $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$;
(7) $l\boldsymbol{A} + m\boldsymbol{A} = (l + m)\boldsymbol{A}$;
(8) $l(\boldsymbol{A} + \boldsymbol{B}) = l\boldsymbol{A} + l\boldsymbol{B}$.§

II. There exist n vectors of the first order, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$, such that any linear relation, $l_1\boldsymbol{e}_1 + l_2\boldsymbol{e}_2 + \cdots + l_n\boldsymbol{e}_n = 0$, implies $l_1 = 0, l_2 = 0, \cdots, l_n = 0$.

Definition. k vectors of any order $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_k$ are said to be *independent* if a linear relation, $l_1\boldsymbol{A}_1 + l_2\boldsymbol{A}_2 + \cdots + l_k\boldsymbol{A}_k = 0$, implies $l_1 = 0, l_2 = 0, \cdots, l_k = 0$.

III. If $\boldsymbol{A}^{(i)}$ is a vector of the i-th order, and if $\boldsymbol{B}^{(j)}$ and $\boldsymbol{C}^{(j)}$ are vectors of the j-th order $(i + j \leqq n)$, then

(1) $[\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}]$ is a uniquely determined vector of the $(i+j)$-th order;
(2) $[\boldsymbol{A}^{(i)}, (\boldsymbol{B}^{(j)} + \boldsymbol{C}^{(j)})] = [\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}] + [\boldsymbol{A}^{(i)}, \boldsymbol{C}^{(j)}]$.

* Ibid., p. 133, no. 173.

† Ibid., p. 135, no. 175.

‡ This system of axioms is compiled from GRASSMANN; all, however, are not given explicitly by him. Some are given as theorems; others are implied in his definitions. The system, as such, is new.

§ The property $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ can be proved by showing that $(\boldsymbol{A} + \boldsymbol{B}) - (\boldsymbol{B} + \boldsymbol{A}) = 0$, by use of (6), (7), (8); hence by (5), the result follows.

Definition. The expression $[\boldsymbol{A}^{(i)}, \boldsymbol{B}^{(j)}]$ is called the outer product of $\boldsymbol{A}^{(i)}$ and $\boldsymbol{B}^{(j)}$.

IV. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of the first order

$$[l\boldsymbol{a}, m\boldsymbol{b}] = lm[\boldsymbol{a}, \boldsymbol{b}]; \tag{1}$$

$$[\boldsymbol{a}, \boldsymbol{b}] + [\boldsymbol{b}, \boldsymbol{a}] = 0. \tag{2}$$

V. If $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i$ and $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j$ are vectors of the first order and if $i + j \leqq n$, then

$$([\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i], [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j]) = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_i, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_j].$$

VI. n independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of the first order exist, such that $[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n] = 1$.

Definition. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ be any n independent vectors of the first order; the products involving the different combinations of these vectors i at a time are called the *multiplicative combinations* of the i-th order of these vectors.

VII. If $\boldsymbol{A}^{(i)}$ is any vector of the i-th order, then n independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of the first order and

$$N_i = \frac{n!}{i!\,(n-i)!}$$

numbers, $l_1, l_2, \cdots, l_{N_i}$, exist such that $\boldsymbol{A}^{(i)} = l_1\boldsymbol{E}_1 + l_2\boldsymbol{E}_2 + \cdots + l_{N_i}\boldsymbol{E}_{N_i}$, where the $\boldsymbol{E}$'s are the multiplicative combinations of the i-th order of $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$.

Complements, regressive and inner products are defined for n dimensions precisely as on p. 457 for three dimensions. Applying the definition of inner multiplication to the units we h ve

$$[\boldsymbol{e}_i | \boldsymbol{e}_j] = \begin{matrix} 0 & (i \neq j)^* \\ 1 & (i = j), \end{matrix}$$

where the units are the $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of Axiom VI.

IV. Interpretation of Symbolic Forms and Comparison with the Axioms.

10. *Definitions.* It will now be shown that symbolic differential parameters of the special form

$$dx^2 + dy^2 + dz^2 = (f_{(1)}dx + f_{(2)}dy + f_{(3)}dz)^2 \tag{0}$$

may be regarded as vectors of a three-dimensional space by interpreting $f_{(1)}$, $f_{(2)}$, $f_{(3)}$ as unit vectors along the x-, y-, and z-axes respectively. This will be done by showing that with suitable definition of outer product, etc., they satisfy the Axioms I–VII for $n = 3$.

Linear functions of any number of symbolic parameters of the type (f, a, b), where f is a symbolic function of the differential form (30), will be shown to be

* Compare with the formulas for $f_{(i)}f_{(j)}$, footnote, p. 450.

one-dimensional vectors; it should be noted that $f_{(1)}, f_{(2)}, f_{(3)}$ are themselves of this type, since $f_{(1)} = (f, y, z), f_{(2)} = (f, z, x), f_{(3)} = (f, x, y)$.

Also $(f, \phi, \psi)(\phi, \psi, a)$ is a linear function of parameters of this type since, when it is expanded, ϕ and ψ disappear according to equations in the footnote, p. 450.

Linear functions of parameters of the type (f, ϕ, a) involving two symbolic functions will be shown to be vectors of the second order.

For effecting transformations of certain symbolic expressions, the following determinant theorem, stated for determinants of the third order, is found useful.

Denote

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by $|a, b, c|$; then

$$|a,b,c|\cdot|d,e,f| = |d,b,c|\cdot|a,e,f| + |e,b,c|\cdot|d,a,f| + |f,b,c|\cdot|d,e,a|.*$$

Let us use the notation

$$\boldsymbol{a} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, a) = a_{(1)}f_{(1)} + a_{(2)}f_{(2)} + a_{(3)}f_{(3)},$$
$$\boldsymbol{b} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, b) = b_{(1)}f_{(1)} + b_{(2)}f_{(2)} + b_{(3)}f_{(3)},$$
$$\boldsymbol{c} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, c) = c_{(1)}f_{(1)} + c_{(2)}f_{(2)} + c_{(3)}f_{(3)},$$

where a, b, c are functions of x, y, z.

We define outer products as follows:

$$[\boldsymbol{a}, \boldsymbol{b}] = \frac{1}{2!}(f, \phi, \psi)(f, a, b),$$

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = ([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) = (\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]) = (a, b, c).\dagger$$

We take Axioms $\mathrm{III}_{(2)}$ and $\mathrm{IV}_{(1)}$ as definitions of the expressions

$$[\boldsymbol{A}^{(i)}, (\boldsymbol{B}^{(j)} + \boldsymbol{C}^{(j)})] \qquad \text{and} \qquad [l\boldsymbol{a}, m\boldsymbol{b}].$$

11. *Comparison with the axioms.* We proceed to show that with these definitions Axioms I to VII are satisfied.

* For determinants of the n-th order

$$|a_1, a_2, \cdots, a_n|\cdot|b_1, b_2, \cdots, b_n| = \sum_{i=1}^{n} |b_i, a_2, \cdots, a_n|\cdot|b_1, b_2, \cdots, b_{i-1}, a_1, b_{i+1}, \cdots, b_n|.$$

See MASCHKE, *Differential parameters of the first order*, Transactions of the American Mathematical Society, vol. 7 (1906), p. 70, equation (1). Cf. also E. PASCAL, Lincei Rendiconti, 1888.

† This definition is given in order to conform to the customary definition of $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ which makes this product a number rather than a vector of the third order. A definition more in accord with the other definitions given here would be

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = (f, \phi, \psi)(a, b, c)/1\sqrt{3!}.$$

$I_{(1)} \cdots I_{(8)}$ are satisfied by the fundamental assumptions in regard to symbols.*

To show that II is satisfied we show that $f_{(1)}$, $f_{(2)}$, $f_{(3)}$ are linearly independent. Assume then a linear relation

$$pf_{(1)} + qf_{(2)} + rf_{(3)} = 0;$$

multiplying through by $f_{(1)}$, we have

$$pf_{(1)}^2 + qf_{(1)}f_{(2)} + rf_{(1)}f_{(3)} = 0.$$

But $f_{(1)}^2 = 1$, $f_{(1)}f_{(2)} = 0$, $f_{(1)}f_{(3)} = 0$. Hence the relation reduces to $p = 0$. Similarly, multiplying in turn by $f_{(2)}$, $f_{(3)}$ we find $q = 0$, $r = 0$. Hence $f_{(1)}$, $f_{(2)}$, and $f_{(3)}$ may be taken as the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of II.

$III_{(2)}$ and $IV_{(1)}$ are satisfied by definition.

$III_{(1)}$ is satisfied by definition in case the two factors are of the first order; also in case the factors are one of the second order and one of the first order in the forms

$$\frac{1}{\sqrt{2}}(f, \phi, \psi)(f, a, b)$$

and $(f, \phi, \psi)(\phi, \psi, a)/2$ respectively. There remains the case in which the factors are of the form (ϕ, ψ, a) and $(f, \phi, \psi)(\phi, \psi, b)$. In this case, since $(x, y, z) = 1$, we may write

$$(\phi, \psi, a) = (\phi, \psi, a)(x, y, z) = (\phi, \psi, x)(a, y, z) + (\phi, \psi, y)(a, z, x) + (\phi, \psi, z)(a, x, y)$$

by the determinant theorem. Now

$$(f, \phi, \psi) = (f, \phi, \psi)(x, y, z) = (f, y, z)(\phi, \psi, x) + (f, z, x)(\phi, \psi, y) + (f, x, y)(\phi, \psi, z).$$

Hence

$$(f, \phi, \psi)(f, y, z) = (f, y, z)^2(\phi, \psi, x) + (f, z, x)(f, y, z)(\phi, \psi, y) + (f, x, y)(f, y, z)(\phi, \psi, z)$$

$$= f_{(1)}^2(\phi, \psi, x) + f_{(1)}f_{(2)}(\phi, \psi, y) + f_{(1)}f_{(3)}(\phi, \psi, z) = (\phi, \psi, x);$$

and similarly

$$(f, \phi, \psi)(f, z, x) = (\phi, \psi, y),$$

$$(f, \phi, \psi)(f, x, y) = (\phi, \psi, z).$$

Hence finally

$$(\phi, \psi, a) = (a, y, z) \cdot (f, \phi, \psi)(f, y, z) + (a, z, x) \cdot (f, \phi, \psi)(f, z, x) + (a, x, y) \cdot (f, \phi, \psi)(f, x, y).$$

Hence if

$$\boldsymbol{A} = \frac{1}{\sqrt{2}}(\phi, \psi, a) \qquad \text{and} \qquad \boldsymbol{B} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, b),$$

*See p. 451.

then

$$[\boldsymbol{A}, \boldsymbol{B}] = (a, y, z)(b, y, z) + (a, z, x)(b, z, x) + (a, x, y)(b, x, y).$$

$\mathrm{IV}_{(2)}$ is satisfied by a property of determinants.

V is satisfied by definition.

VI is satisfied by taking a, b, c, equal to x, y, z, respectively.

We shall prove that Axiom VII is satisfied by showing that every symbolic differential parameter of the type (f, a, b) can be expressed linearly in terms of (f, y, z), (f, z, x), (f, x, y) and that every symbolic differential parameter of the type (f, ϕ, a) can be expressed linearly in terms of the three parameters (f, ϕ, x), (f, ϕ, y), (f, ϕ, z).

Using the determinant theorem we have

$$(f, a, b) = (f, a, b)(x, y, z) = (x, a, b)(f, y, z) + (y, a, b)(f, z, x) + (z, a, b)(f, x, y);$$

and also

$$(f, \phi, a) = (f, \phi, a)(x, y, z) = (a, y, z)(f, \phi, x) + (a, z, x)(f, \phi, y) + (a, x, y)(f, \phi, z).$$

It is shown in the next article that $(f, \phi, x)/\sqrt{2}$, $(f, \phi, y)/\sqrt{2}$, $(f, \phi, z)/\sqrt{2}$ are the multiplicative combinations of the second order of $\boldsymbol{e}_{(1)} = f_{(1)}$, $\boldsymbol{e}_{(2)} = f_{(2)}$, $\boldsymbol{e}_{(3)} = f_{(3)}$. Axiom VII is, therefore, satisfied if we take

$$(f, y, z) = f_{(1)}, \qquad (f, z, x) = f_{(2)}, \qquad (f, x, y) = f_{(3)}$$

as the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

The statement made at the head of this section is therefore justified.

The similar work for n dimensions is an easy generalization; hence we have

THEOREM III. *The symbols $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$, of the special differential form*

$$\sum_{i=1}^{n} dx_1^2,$$

may be regarded as unit vectors along the coördinate axes; linear functions of parameters of the type $(f_1, f_2, \cdots, f_k, a_1, a_2, \cdots, a_{n-k})$ involving k symbolic functions $f_1, \cdots, f_k$, may then be interpreted as k-dimensional vectors lying in the space determined by $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$.

In n dimensions we use the notation

$$\boldsymbol{a}_i = \frac{1}{(n-1)!}(\phi_1, \cdots, \phi_{n-1}, f)(\phi_1, \cdots, \phi_{n-1}, a_i) = \sum_{j=1}^{n} a_{i(j)} f_{(j)},$$

$$\boldsymbol{b} = \frac{1}{(n-1)!}(\phi_1, \cdots, \phi_{n-1}, f)(\phi_1, \cdots, \phi_{n-1}, b_i),$$

and define outer products as follows:

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k] = \frac{1}{(n-k)!\,\sqrt{k!}}(\phi_1, \cdots, \phi_{n-k}, f_1, \cdots, f_k)(\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k),$$

$$([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_j]) = \frac{1}{(n-k-j)!\,\sqrt{(k+j)!}}$$

$$\times(\phi_1, \cdots, \phi_{n-k-j}, f_1, \cdots, f_{k+j})(\phi_1, \cdots, \phi_{n-k-j}, a_1, \cdots, a_k, b_1, \cdots, b_j)$$

$$(k+j<n).$$

V. Formulas and Applications.

12. *General formulas.* The symbolic expressions for various combinations of vectors will now be obtained and the formulas will then be applied to obtain new relations connecting symbolic differential parameters. We first give different symbolic expressions for the units and their multiplicative combinations.

$$\boldsymbol{e}_1 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = (f, y, z),$$

$$\boldsymbol{e}_2 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, y) = (f, z, x), \tag{31}$$

$$\boldsymbol{e}_3 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, z) = (f, x, y).$$

For,

$$(f, \phi, \psi) = (f, \phi, \psi)(x, y, z)$$

$$= (f, y, z)(\phi, \psi, x) + (f, z, x)(\phi, \psi, y) + (f, x, y)(\phi, \psi, z)$$

and

$$\tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = \tfrac{1}{2}\{2(f, y, z)\} = (f, y, z),$$

since

$$(\phi, \psi, x)(\phi, \psi, x) = 2$$

and

$$(\phi, \psi, x)(\phi, \psi, y) = (\phi, \psi, x)(\phi, \psi, z) = 0.$$

Similarly for the others.

By definition

$$\boldsymbol{e}_1 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, x).$$

For the units of the second order we have

$$[\boldsymbol{e}_2, \boldsymbol{e}_3] = |\,\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z) = \frac{1}{\sqrt{2}}(\phi, \psi, x),$$

$$[\boldsymbol{e}_3, \boldsymbol{e}_1] = |\,\boldsymbol{e}_2 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, z, x) = \frac{1}{\sqrt{2}}(\phi, \psi, y), \tag{32}$$

$$[\boldsymbol{e}_1, \boldsymbol{e}_2] = |\,\boldsymbol{e}_3 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, x, y) = \frac{1}{\sqrt{2}}(\phi, \psi, z).$$

For, by definition, $|\,\boldsymbol{e}_1 = [\boldsymbol{e}_2, \boldsymbol{e}_3]$, where $\boldsymbol{e}_2 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, y)$ and $\boldsymbol{e}_3 = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, z)$ and by definition, p. 462,

$$[\boldsymbol{e}_2, \boldsymbol{e}_3] = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z);$$

that $(f, \phi, \psi)(f, y, z) = (\phi, \psi, x)$ is proved by writing

$$(f, \phi, \psi) = (f, \phi, \psi)(x, y, z),$$

etc., as in the proof of formulas (31).

By this same method it may be shown that

(33) $\boldsymbol{a} = \frac{1}{2}(f, \phi, \psi)(\phi, \psi, a) = (a, y, z)\boldsymbol{e}_1 + (a, z, x)\boldsymbol{e}_2 + (a, x, y)\boldsymbol{e}_3;$

for $(\phi, \psi, a) = (\phi, \psi, a)(x, y, z) = (\phi, \psi, x)(a, y, z)$, etc., and $\frac{1}{2}(f, \phi, \psi)(\phi, \psi, x) = \boldsymbol{e}_1$, etc.

Other important formulas follow:

(34) $$|\boldsymbol{a} = \frac{1}{\sqrt{2}}(\phi, \psi, a);^*$$

for by (33)

$$|\boldsymbol{a} = (a, y, z)|\boldsymbol{e}_1 + (a, z, x)|\boldsymbol{e}_2 + (a, x, y)|\boldsymbol{e}_3$$

$$= \frac{1}{\sqrt{2}}\{(a, y, z)(\phi, \psi, x) + (a, z, x)(\phi, \psi, y) + (a, x, y)(\phi, \psi, z)\},$$

and

$$\frac{1}{\sqrt{2}}(\phi, \psi, a) = \frac{1}{\sqrt{2}}(\phi, \psi, a)(x, y, z) = \frac{1}{\sqrt{2}}(a, y, z)(\phi, \psi, x),$$

etc., as above.

(35) $$[\boldsymbol{a}|\boldsymbol{b}] = \tfrac{1}{2}(f, \phi, a)(f, \phi, b) = (a, y, z)(b, y, z) + (a, z, x)(b, z, x) + (a, x, y)(b, x, y) = a_{(1)}b_{(1)} + a_{(2)}b_{(2)} + a_{(3)}b_{(3)}.$$

Proof. By (33)

$$\boldsymbol{b} = (b, y, z)\boldsymbol{e}_1 + (b, z, x)\boldsymbol{e}_2 + (b, x, y)\boldsymbol{e}_3;$$

hence

$$[\boldsymbol{a}|\boldsymbol{b}] = (b, y, z)[\boldsymbol{a}|\boldsymbol{e}_1] + (b, z, x)[\boldsymbol{a}|\boldsymbol{e}_2] + (b, x, y)[\boldsymbol{a}|\boldsymbol{e}_3)$$

and by definition, p. 462,

$$[\boldsymbol{a}|\boldsymbol{e}_1] = (a, y, z),$$

etc., where

$$\boldsymbol{a} = \tfrac{1}{2}(f, \phi, \psi)(\phi, \psi, a) \quad \text{and} \quad |\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(f, \phi, \psi)(f, y, z);$$

hence

$$[\boldsymbol{a}|\boldsymbol{b}] = (a, y, z)(b, y, z) + (a, z, x)(b, z, x) + (a, x, y)(b, x, y).$$

* Except for a numerical factor the complement of any vector expressed in symbolic form is obtained by multiplying by a factor (f, φ, ψ) made up wholly of symbolic functions, including the symbolic functions which occur only once in the expression for the vector, or by leaving out such a factor in case it is present.

Again

$$\begin{aligned}(f,\phi,a)(f,\phi,b) &= \{(a,y,z)(f,\phi,x)+(a,z,x)(f,\phi,y)\\ &\quad +(a,x,y)(f,\phi,z)\}\{(b,y,z)(f,\phi,x)\\ &\quad +(b,z,x)(f,\phi,y)+(b,x,y)(f,\phi,z)\}\\ &= 2\{(a,y,z)(b,y,z)+(a,z,x)(b,z,x)\\ &\quad +(a,x,y)(b,x,y)\}\end{aligned}$$

and this proves formula (35).

$$[\boldsymbol{a},\boldsymbol{b}]=\frac{1}{\sqrt{2}}(f,\phi,\psi)(f,a,b) \tag{36}$$

$$=(x,a,b)[\boldsymbol{e}_2,\boldsymbol{e}_3]+(y,a,b)[\boldsymbol{e}_3,\boldsymbol{e}_1]+(z,a,b)[\boldsymbol{e}_1,\boldsymbol{e}_2].$$

The proof is similar to the proof of (33).

$$|[\boldsymbol{a},\boldsymbol{b}]=[|\boldsymbol{a}|\boldsymbol{b}]=(f,a,b); \tag{37}$$

for from (36)

$$|[\boldsymbol{a},\boldsymbol{b}]=(x,a,b)\boldsymbol{e}_1+(y,a,b)\boldsymbol{e}_2+(z,a,b)\boldsymbol{e}_3=(x,a,b)(f,y,z)$$

$$+(y,a,b)(f,z,x)+(z,a,b)(f,x,y)=(f,a,b).$$

$$([\boldsymbol{a},\boldsymbol{b}],[\boldsymbol{c},\boldsymbol{d}])=(f,\phi,\psi)(\phi,a,b)(\psi,c,d). \tag{38}$$

Proof. By definition,

$$|([\boldsymbol{a},\boldsymbol{b}],[\boldsymbol{c},\boldsymbol{d}])=(|[\boldsymbol{a},\boldsymbol{b}]|[\boldsymbol{c},\boldsymbol{d}])=[\Sigma(x,a,b)\boldsymbol{e}_1,\Sigma(x,c,d)\boldsymbol{e}_1]$$

$$=\begin{vmatrix}(x,a,b),(x,c,d)\\(y,a,b),(y,c,d)\end{vmatrix}[\boldsymbol{e}_1,\boldsymbol{e}_2]+\begin{vmatrix}(y,a,b),(y,c,d)\\(z,a,b),(z,c,d)\end{vmatrix}[\boldsymbol{e}_2,\boldsymbol{e}_3]$$

$$+\begin{vmatrix}(z,a,b),(z,c,d)\\(x,a,b),(x,c,d)\end{vmatrix}[\boldsymbol{e}_3,\boldsymbol{e}_1];$$

again,

$$(f,\phi,\psi)(\phi,a,b)(\psi,c,d)=(\phi,a,b)\{(f,\phi,x)(\psi,y,z)(\psi,c,d)$$

$$+(f,\phi,y)(\psi,z,x)(\psi,c,d)+(f,\phi,z)(\psi,x,y)(\psi,c,d)\}$$

$$=(\phi,a,b)\{(f,\phi,x)(x,c,d)+(f,\phi,y)(y,c,d)+(f,\phi,z)(z,c,d)\}.$$

Now

$$(f,\phi,x)(\phi,a,b)=(f,y,x)(\phi,z,x)(\phi,a,b)$$

$$+(f,z,x)(\phi,x,y)(\phi,a,b)=(y,a,b)(f,y,x)+(z,a,b)(f,z,x),$$

with similar expressions for $(f,\phi,y)(\phi,a,b)$ and $(f,\phi,z)(\phi,a,b)$. Sub-

stituting, we have finally

$$(f, \phi, \psi)(\phi, a, b)(\psi, c, d) = \{(y, a, b)(f, y, x) + (z, a, b)(f, z, x)\}(x, c, d) + \{(z, a, b)(f, z, y) + (x, a, b)(f, x, y)\}(y, c, d) + \{(x, a, b)(f, x, z) + (y, a, b)(f, y, z)\}(z, c, d)$$

$$= (f, x, y)\begin{vmatrix}(x, a, b), (x, c, d)\\ (y, a, b), (y, c, d)\end{vmatrix} + (f, y, z)\begin{vmatrix}(y, a, b), (y, c, d)\\ (z, a, b), (z, c, d)\end{vmatrix} + (f, z, x)\begin{vmatrix}(z, a, b), (z, c, d)\\ (x, a, b), (x, c, d)\end{vmatrix}.$$

The expression found above for $|([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{c}, \boldsymbol{d}])$ is clearly the complement of this final expression for $(f, \phi, \psi)(\phi, a, b)(\psi, c, d)$; hence the formula is proved.

(39) $$([\boldsymbol{a}, \boldsymbol{b}]\,|\,\boldsymbol{c}) = (f, \phi, c)(\phi, a, b).$$

For by (33) or (34)

$$|\boldsymbol{c} = \Sigma(c, y, z)[\boldsymbol{e}_2, \boldsymbol{e}_3];$$

hence

$$([\boldsymbol{a}, \boldsymbol{b}]\,|\,\boldsymbol{c}) = (c, y, z)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_2, \boldsymbol{e}_3]) + (c, z, x)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_3, \boldsymbol{e}_1]) + (c, x, y)([\boldsymbol{a}, \boldsymbol{b}], [\boldsymbol{e}_1, \boldsymbol{e}_2])$$

$$= (\phi, a, b)\Sigma(c, y, z)(f, \phi, \psi)(\psi, y, z) \qquad \text{by (38)},$$

$$= (\phi, a, b)\Sigma(f, \phi, x)(c, y, z) = (f, \phi, c)(\phi, a, b).$$

(40) $$([\boldsymbol{a}, \boldsymbol{b}]\,|\,[\boldsymbol{c}, \boldsymbol{d}]) = (f, a, b)(f, c, d).^*$$

For, by formula (37), $|[\boldsymbol{c}, \boldsymbol{d}] = \Sigma(x, c, d)\boldsymbol{e}_1$; hence

$$([\boldsymbol{a}, \boldsymbol{b}]\,|\,[\boldsymbol{c}, \boldsymbol{d}]) = (x, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_1] + (y, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_2] + (z, c, d)[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}_3]$$

$$= (x, c, d)(x, a, b) + (y, c, d)(y, a, b) + (z, c, d)(z, a, b);$$

and $(f, a, b)(f, c, d)$ expanded gives the same.

We give below the extensions of the formulas (31)–(40) to n dimensions; the proofs are similar to the proofs given above for three dimensions.

(31_n) $$\boldsymbol{e}_i = \frac{1}{(n-1)!}(f, \phi_1, \cdots, \phi_{n-1})(\phi_1, \cdots, \phi_{n-1}, x_i)$$

$$= (x_1, \cdots, x_{i-1}, f, x_{i+1}, \cdots, x_n).$$

* Except for a numerical factor the inner product of two vectors in symbolic notation is obtained by writing them both (or their complements) in terms of the same symbolic functions as far as possible.

$$(32_n)\qquad [\boldsymbol{e}_{c_1}, \cdots, \boldsymbol{e}_{c_k}] = \frac{1}{(n-k)!\sqrt{k!}} F_1 F_2 = \frac{1}{\sqrt{k!}}(f_1, \cdots, f_k, x_{c_{k+1}}, \cdots, x_{c_n}),$$

$$F_1 = (f_1, \cdots, f_k, \phi_1, \cdots, \phi_{n-k}), \qquad F_2 = (\phi_1, \cdots, \phi_{n-k}, x_{c_1}, \cdots, x_{c_k}),$$

where $c_1, \cdots, c_k$ stands for any combination of k of the subscripts $1, 2, \cdots, n$, and $c_{k+1}, \cdots, c_n$ are the remaining subscripts in such order that $(x_{c_1}, x_{c_2}, \cdots, x_{c_n}) = +1$.

$$(33_n)\qquad \boldsymbol{a} = \frac{1}{(n-1)!}(f, \phi_1, \cdots, \phi_{n-1})(\phi_1, \cdots, \phi_{n-1} a) \sum_{i=1}^{n} (x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n)\boldsymbol{e}_i.$$

$$(34_n)\qquad |\boldsymbol{a} = \frac{1}{\sqrt{(n-1)!}}(f_1, \cdots, f_{n-1}, a).$$

$$(35_n)\qquad [\boldsymbol{a}|\boldsymbol{b}] = \frac{1}{(n-1)!}(f_1, \cdots, f_{n-1}, a)(f_1, \cdots, f_{n-1}, b).$$

$$(36_n)\qquad [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k] = \frac{1}{(n-k)!\sqrt{k!}}(\phi_1, \cdots, \phi_{n-k}, f_1, \cdots, f_k)(\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k)$$

$$= \sum_c (a_1, \cdots, a_k, x_{c_1}, \cdots, x_{c_{n-k}})[\boldsymbol{e}_{c_{n-k+1}}, \cdots, \boldsymbol{e}_{c_n}].$$

$$(37_n)\qquad |[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k] = (|[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_i]|[\boldsymbol{a}_{i+1}, \cdots, \boldsymbol{a}_k]) = \frac{1}{\sqrt{(n-k)!}}(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k).$$

If $k + j > n$,

$$([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k], [\boldsymbol{b}_1, \cdots, \boldsymbol{b}_j]) = \frac{1}{(n-k)!\ (n-j)!\ \sqrt{(k+j-n)!}} M_1 M_2,$$

$$(38_n)\qquad M_1 = (\phi_1, \cdots, \phi_{n-k}, \psi_1, \cdots, \psi_{n-j}, f_1, \cdots, f_{k+j-n}),$$

$$M_2 = (\phi_1, \cdots, \phi_{n-k}, a_1, \cdots, a_k)(\psi_1, \cdots, \psi_{n-j}, b_1, \cdots, b_j).$$

$$(39_n)\qquad ([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k+r}]\,|\,[\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r]) = \frac{1}{(n-k-r)!\ \sqrt{k!}} M_3,$$

$$M_3 = (f_1, \cdots, f_k, \phi_{k+1}, \cdots, \phi_{n-r}, b_1, \cdots, b_r)(\phi_{k+1}, \cdots, \phi_{n-r}, a_1, \cdots, a_{k+r}),$$

$$([\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r]\,|\,[\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r])$$

$$(40_n)\qquad = \frac{1}{(n-r)!}(\phi_1, \cdots, \phi_{n-r}, a_1, \cdots, a_r)(\phi_1, \cdots, \phi_{n-r}, b_1, \cdots, b_r).$$

13. *Application to the symbolic invariant theory.* Results of the Grassmann theory may be employed to derive formulas in the symbolic theory. Three examples of this application are given below.*

1. From formula (27_n) follows by use of formula (38_n)

$$
\begin{aligned}
&(f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, \psi_1, \cdots, \psi_{k+j-1})(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k) \\
&\quad \times (\phi_1, \cdots, \phi_{n-j}, b_1, \cdots, b_j) = \frac{(n-k)!\ (n-j)!}{(2n-k-j)!} \\
&\quad \times \sum_c (f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, \psi_1, \cdots, \psi_{k+j-n}) \\
&\quad \times (f_1, \cdots, f_{n-k}, \phi_1, \cdots, \phi_{n-j}, b_{c_1}, \cdots, b_{c_{k+j-n}})(a_1, \cdots, a_k, b_{c_1'}, \cdots, b_{c_{n-k}'}).\dagger
\end{aligned} \tag{41}
$$

2. From formula (28_n) follows by use of formula (39_n)

$$
\begin{aligned}
&(f_1, \cdots, f_k, \phi_{k+1}, \cdots, \phi_{n-r}, b_1, \cdots, b_r)(\phi_{k+1}, \cdots, \phi_{n-r}, a_1, \cdots, a_{k+r}) \\
&= \frac{(n-k-r)!}{(n-k)!\ (n-r)!} \sum_c (f_1, \cdots, f_{n-r}, a_{c_1}, \cdots, a_{c_r})(f_1, \cdots, f_{n-r}, b_1, \cdots, b_r) \\
&\qquad \times (\phi_1, \cdots, \phi_{n-k}, a_{c_1'}, \cdots, a_{c_k'})(\phi_1, \cdots, \phi_{n-k}, \psi_1, \cdots, \psi_k).
\end{aligned} \tag{42}
$$

3. From formula (29_n) follows by use of formulas (35_n) and (40_n)

$$
\begin{aligned}
&\frac{1}{(n-k)!}(f_1, \cdots, f_{n-k}, a_1, \cdots, a_k)(f_1, \cdots, f_{n-k}, b_1, \cdots, b_k) \\
&= \frac{1}{\{(n-1)!\}^k} \begin{vmatrix} (f_1, a_1)(f_1, b_1), & (f_1, a_1)(f_1, b_2), & \cdots, & (f_1, a_1)(f_1, b_k) \\ (f_2, a_2)(f_2, b_1), & (f_2, a_2)(f_2, b_2), & \cdots, & (f_2, a_2)(f_2, b_k) \\ \cdots & \cdots & \cdots & \cdots \\ (f_k, a_k)(f_k, b_1), & (f_k, a_k)(f_k, b_2), & \cdots, & (f_k, a_k)(f_k, b_k) \end{vmatrix},
\end{aligned} \tag{43}
$$

where $(f_i, a_i) = (f_{i1}, f_{i2}, \cdots, f_{in-1}, a_i)$ and f_{ij} is a symbolic function equivalent to f.

These formulas can be shown to hold for the general differential form.

14. *Applications to geometry.* As heretofore, we regard the variables $x_1, x_2, \cdots, x_n$ as cartesian coördinates of a point in an n-dimensional space; we wish to determine how the k-dimensional vector $(f_1, \cdots, f_k, a_1, \cdots, a_{n-k})$ is related to the surfaces $a_1 = \text{const.}, a_2 = \text{const.}, \cdots, a_{n-k} = \text{const.}$

First we show that the $(n-1)$-dimensional vector $(f_1, \cdots, f_{n-1}, a)$ is, at

* Formulas (41) and (42) are new.

For formula (43) see MASCHKE, *Differential parameters of the first order*, loc. cit., p. 73.

† $c_1, \cdots, c_{k+j-n}$ stands for any combination of $k+j-n$ of the subscripts, 1, 2, $\cdots, j$; $c_1', \cdots, c_{n-k}'$ are the remaining subscripts. In order to obtain the correct sign in each term, the order of the subscripts must be such that $c_1, \cdots, c_{k+j-n}, c_1', \cdots, c_{n-k}'$ may be obtained from 1, 2, $\cdots, j$, by an even number of transpositions.

each point of space, tangent to one of the surfaces $a = \text{const.}$; for when a is linear, by formulas (33_n) and (34_n),

$$\frac{1}{\sqrt{(n-1)!}}(f_1, \cdots, f_{n-1}, a) = \sum_{i=1}^{n} a_{(i)} e_i,$$

is parallel to the planes $a = \text{const.}$

When a is not linear, then the linear space of $n-1$ dimensions, tangent at a given point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ to the surface $a = \text{const.}$ which passes through that point, is given by

$$L \equiv \sum_{i=1}^{n} (a_{(i)})_{x=\bar{x}} x_i = \text{const.},$$

and clearly

$$(f_1, \cdots, f_{n-1}, a)_{x=\bar{x}} = (f_1, \cdots, f_{n-1}, L).$$

The linear space tangent to the intersection of $a_1 = \text{const.}$, $a_2 = \text{const.}$ will be the space common to the linear spaces tangent to the two hyper-surfaces. A vector lying in this space is found by taking the outer product of the vectors $(f_1, \cdots, f_{n-1}, a_1)$ and $(f_1, \cdots, f_{n-1}, a_2)$. By formula (37_n) this product may be written, except for a numerical factor, $(f_1, \cdots, f_{n-2}, a_1, a_2)$.

This argument can be repeated for three functions a_1, a_2, a_3, and again, for any number, $a_1, a_2, \cdots, a_{n-k}$; hence we have:

THEOREM IV. *If the derivatives $f_{(1)}, f_{(2)}, \cdots, f_{(n)}$ of a symbolic function of the differential form*

$$\sum_{i=1}^{n} dx_i^2$$

are interpreted as unit vectors along the coördinate axes, then

$$(f_1, f_2, \cdots, f_k, a_1, a_2, \cdots, a_{n-k})$$

may be interpreted as a k-dimensional vector, tangent to the k-dimensional spread which is common to $a_1 = \text{const.}$, $a_2 = \text{const.}$, $\cdots$, $a_{n-k} = \text{const.}$

VI. THE GENERAL DIFFERENTIAL FORM.

15. *Change of variables in three dimensions.* So far we have been considering differential parameters of a special type of differential form: namely, the differential form for length of arc in an Euclidean space; i. e., a form of the type

$$\sum_{i=1}^{n} dx_i^2,$$

or any form into which such a form can be transformed.

In this section it is shown that Theorem IV can be extended to apply to any space which can be considered a part of an Euclidean space (of higher dimensions); for example, to a surface lying in three dimensions.

Let us consider in an ordinary space of three dimensions the transformation

$$u = u(x, y, z), \qquad v = v(x, y, z), \qquad w = w(x, y, z).$$

We can find dx, dy, dz, in terms of du, dv, dw, by solving the equations

$$du = u_{(1)}dx + u_{(2)}dy + u_{(3)}dz,$$
$$dv = v_{(1)}dx + v_{(2)}dy + v_{(3)}dz,$$
$$dw = w_{(1)}dx + w_{(2)}dy + w_{(3)}dz.$$

If the resulting values are substituted in

$$ds = f_{(1)}dx + f_{(2)}dy + f_{(3)}dz$$

we obtain

$$ds = \frac{(f, v, w)}{(u, v, w)}du + \frac{(u, f, w)}{(u, v, w)}dv + \frac{(u, v, f)}{(u, v, w)}dw.$$

By Theorem IV the coefficients* of du, dv, dw may be regarded as linear vectors tangent to the intersections of $v = \text{const.}$, $w = \text{const.}$; $w = \text{const.}$, $u = \text{const.}$; and $u = \text{const.}$, $v = \text{const.}$, respectively. Let us denote these coefficients by $t_{(1)}$, $t_{(2)}$, $t_{(3)}$. It can be shown precisely as in the two-dimensional case that $t_{(1)}$, $t_{(2)}$, $t_{(3)}$ are the derivatives with respect to u, v, w of the function into which the symbolic function f transforms. Hence the transformed function $t(u, v, w)$ is a symbolic function of the differential quantic into which $dx^2 + dy^2 + dz^2$ transforms and the derivatives $t_{(u)}$, $t_{(v)}$, $t_{(w)}$ (i. e., $t_{(1)}$, $t_{(2)}$, $t_{(3)}$) may be interpreted as vector functions of position, tangent at each point of space to the parameter curves which pass through that point. The transformed differential form is given by $ds^2 = df^2 = dt^2 = (t_{(1)}du + t_{(2)}dv + t_{(3)}dw)^2$ where $t_{(1)} = (f, v, w)/(u, v, w)$, etc.

16. *The general two-dimensional case.* On the surface $w = a$ (a definite const.), $dw = 0$. In our formula for ds let us write a for w to indicate that only such values of x, y, z are to be used as will satisfy the equation $w = a$.

We have, then, on the surface

$$ds^2 = (t_{(1)}du + t_{(2)}dv)^2 = Edu^2 + 2Fdudv + Gdv^2$$

where

$$t_{(1)}^2 = E, \qquad t_{(1)}t_{(2)} = F, \qquad t_{(2)}^2 = G.\dagger$$

This gives rise to a vector system *on a surface*, having *variable units* $t_{(1)}$, $t_{(2)}$ which are tangent to the parameter curves of the surface. The usual vector expressions, as inner and outer products, and complements can be set up in terms

*Observe that the common denominator of these coefficients is an ordinary number, being the Jacobian of the functions u, v, w with respect to x, y, z.

†The function $f = f_{(1)} \cdot x + f_{(2)} \cdot y + f_{(3)} \cdot z$ represents the vector from the origin to the point x, y, z; hence $t(u, v)$ represents the vector from the origin to any point on the surface $w = a$.

of these variable units; these expressions take the same form as the corresponding expressions in the plane given on p. 453.

It should be noticed that here, since $w = a$ is an arbitrary surface, we are considering a perfectly general binary quadratic differential form; hence

THEOREM V. *If* $A \equiv Edu^2 + 2Fdudv + Gdv^2$ *is any binary quadratic differential form, the partial derivatives* $\partial t/\partial u$, $\partial t/\partial v$ *of a symbolic function of* A *may be interpreted as vectors, of length* $\sqrt{E}$, $\sqrt{G}$ *respectively, tangent to the parameter curves of the surface characterized by* A.

17. *The general k-dimensional case.* In the same way we consider in an Euclidean space of n dimensions, the sub-space of k dimensions determined by $a_1 = \text{const.}, \cdots, a_{n-k} = \text{const.}$ For arc length in the sub-space we shall have $ds^2 = (\Sigma t_{(i)} du_i)^2$, where

$$t_{(i)} = \frac{(u_1, u_2, \cdots, u_{i-1}, f, u_{i+1}, \cdots, u_k, a_1, \cdots, a_{n-k})}{(u_1, \cdots, u_k, a_1, \cdots, a_{n-k})}$$

and where $t(u_1, \cdots, u_k) = f(x_1, \cdots, x_n)$.

Precisely as in the three-dimensional cases it can be shown that $t_{(i)} = \partial t/\partial u_i$; hence we have

THEOREM VI. *If*

$$A \equiv \sum_{i,j=1}^{k} E_{ij} du_i du_j$$

is the differential form giving length of arc in any space of k *dimensions belonging to an Euclidean space of* n *dimensions, then the partial derivatives* $\partial t/\partial u_1, \cdots, \partial t/\partial u_k$ *of a symbolic function of* A *may be interpreted as vectors tangent to the parameter curves of the sub-space characterized by* A.

VII. SYMBOLIC DIFFERENTIAL PARAMETERS INVOLVING DERIVATIVES OF THE SECOND AND HIGHER ORDERS.

18. *Interpretation of the higher derivatives of the symbols.* Let the space R_n whose arc element is determined by

$$ds^2 = \sum_{i,j=1}^{n} E_{ij} du_i du_j = \left(\sum_{i=1}^{n} f_{(i)} du_i \right)^2$$

be supposed to lie in an Euclidean space S_r of r dimensions. The $f_{(i)}$ may then be interpreted as vectors in S_r tangent to R_n along the parameter curves. Since these tangent vectors lie in S_r, they may be expressed as linear functions of r constant, mutually orthogonal unit vectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r$, in S_r, the coefficients being functions of $u_1, \cdots, u_n$. The derivatives of the $f_{(i)}$ of any order, with respect to the variables $u_1, u_2, \cdots, u_n$, if they do not vanish, are again linear functions of the unit vectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_r$. We have therefore

THEOREM VII. *If the space R_n whose arc element is determined by*

$$ds^2 = \sum_{i,j=1}^{n} E_{ij}\,du_i\,du_j = \left(\sum_{i=1}^{n} f_{(i)}\,du_i\right)^2$$

lies in an Euclidean space S_r of r dimensions, then the derivatives of the symbolic function f, of all orders, with respect to any of the variables $u_1, \cdots, u_n$, may be interpreted as vectors of the space S_r.

Since symbolic differential parameters involving derivatives of the E_{ij} to any order are expressible in terms of derivatives of the symbolic functions, it follows that all such differential parameters may be interpreted as vectors or as combinations of vectors of the space S_r.

19. *Second covariantive derivatives of the symbolic functions.* The second covariantive derivatives of the symbolic functions are given linearly in terms of the ordinary second derivatives and of the first derivatives by the formula

$$f^{(ij)} = f_{(ij)} - \frac{1}{(n-1)!}\phi_{(ij)}(\phi, \psi)(f, \psi)^* \qquad \left(f_{(ij)} = \frac{\partial^2 f}{\partial u_i \partial u_j}\right).$$

It follows that $f^{(ij)}$ may be interpreted as a vector. But we have always $f_{(k)}f^{(ij)} = 0$† which shows that each $f^{(ij)}$ is orthogonal to all the vectors $f_{(k)}$. Hence

THEOREM VIII. *If the $f_{(i)}$ are interpreted as vectors in S_r tangent to R_n along the parameter curves, then the second covariantive derivatives $f^{(ij)}$ may be interpreted as normals to R_n in S_r.*

* MASCHKE, *A symbolic treatment*, etc., loc. cit., p. 457, equation (75).

† MASCHKE, loc. cit., p. 459, equation (84).

SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AND THEIR TRANSFORMATIONS.*

(SECOND MEMOIR)

BY

LUTHER PFAHLER EISENHART

Introduction.

In a former memoir with the same title † we established a transformation of surfaces with isothermal spherical representation of their lines of curvature into surfaces of the same kind, the transformation being such that lines of curvature on a surface and on a transform correspond, and the two surfaces constitute the envelope of a two-parameter family of spheres. This transformation was established by means of a theorem of MOUTARD, concerning partial differential equations of the LAPLACE type with equal invariants, ‡ and with the aid of a transformation of minimal surfaces discovered by THYBAUT. § In our discussion we did not take the equations of the transformation in the form given by THYBAUT, but in the form used by BIANCHI, ‖ in which case the parametric curves on the minimal surfaces are the asymptotic lines. Such a minimal surface and its THYBAUT transform are the focal sheets of a W-congruence, that is, a congruence upon whose focal sheets the asymptotic lines correspond.

The present paper deals with the same transformations obtained by a very different method as a result of which the analysis is much simpler. In § 1 it is shown that when the lines of a W-congruence are subjected to the LIE line-sphere transformation, the congruence of spheres envelope two surfaces upon which the lines of curvature correspond. If S and S_1 denote these two surfaces, and Σ and Σ_1 the two focal sheets of the original W-congruence, the transformation from Σ to Σ_1 carries with it a transformation from S to S_1 without quadrature. In § 1 is determined the characteristic property which Σ must have in order that the lines of curvature on S may have isothermal spherical representation, and the equations of the surfaces are given in simple form. In § 2 the determination of

* Presented to the Society, February 26, 1910.

† Transactions of the American Mathematical Society, vol. 9 (1908), pp. 149–177.

‡ BIANCHI, *Lezioni*, vol. 2, p. 69.

§ *Sur la déformation du paraboloide, etc.*; Annales de l'École Normale (3), vol. 14 (1897), pp. 65–69.

‖ *Lezioni*, vol. 2, pp. 334–338.

surfaces Σ_1 and S_1 of the same kind as Σ and S respectively and standing in the above mentioned relation to them is carried through, and the problem is shown to be reducible to the integration of an ordinary differential equation of the first order and to quadratures.

When S is a minimal surface (cf. § 3), the transformation is that of THYBAUT and the equations take the very form which occurs in his memoir.*

In § 4 it is shown that the transformation of § 2 is identical with that established in the first memoir, and in § 5 we prove the existence of a "theorem of permutability" which shows that the knowledge of two transforms S_1 and S_2 of S carries with it the determination of a surface S' of the same kind, which is a transform of both S_1 and S_2; and, moreover, S' may be obtained without quadrature.

§ 1. *A Surface* Σ *and its Lie transform* S.

The coördinates, ξ, η, ζ, of a surface Σ, referred to its asymptotic lines, may be given by the Lelieuvre formulas †

$$(1)\qquad \frac{\partial \xi}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial \nu_2}{\partial u} & \dfrac{\partial \nu_3}{\partial u} \end{vmatrix}, \qquad \frac{\partial \xi}{\partial v} = -\begin{vmatrix} \nu_2 & \nu_3 \\ \dfrac{\partial \nu_2}{\partial v} & \dfrac{\partial \nu_3}{\partial v} \end{vmatrix},$$

and similar equations for η and ζ, obtained by permuting the quantities ν_1, ν_2, ν_3, which are solutions of an equation of the form

$$(2)\qquad \frac{\partial^2 \theta}{\partial u\, \partial v} = M\theta,$$

where in general M is a function of u and v. Moreover, ν_1, ν_2 and ν_3 are proportional to the direction-cosines of the normal to Σ.

If x, y, z denote the coördinates and X, Y, Z the direction-cosines of the normal to the surface S which is obtained from Σ by the line-sphere transformation of Lie, these quantities are given by ‡

$$(3)\qquad \begin{aligned} x + iy &= -\zeta - \xi\frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & X + iY &= \frac{2\nu_2\xi}{\nu_2 + \nu_3\xi}, \\ x - iy &= \frac{\nu_1 - \nu_3\eta}{\nu_2 + \nu_3\xi}, & X - iY &= \frac{2\nu_3}{\nu_2 + \nu_3\xi}, \\ z &= \frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & Z &= \frac{\nu_3\xi - \nu_2}{\nu_2 + \nu_3\xi}. \end{aligned}$$

From these equations we determine the following values for the fundamental

* L. c.

† EISENHART, *Differential Geometry*, p. 193.

‡ DARBOUX, *Leçons*, vol. 1, p. 249.

quantities of S:

$$E = \frac{\left(\frac{\partial \zeta}{\partial u} - \xi \frac{\partial \eta}{\partial u} + \eta \frac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad F = 0,$$

$$G = -\frac{\left(\frac{\partial \zeta}{\partial v} - \xi \frac{\partial \eta}{\partial v} + \eta \frac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \tag{4}$$

$$D = 2\frac{\partial \xi}{\partial u}\frac{\frac{\partial \zeta}{\partial u} - \xi \frac{\partial \eta}{\partial u} + \eta \frac{\partial \xi}{\partial u}}{(\nu_2 + \nu_3 \xi)^2}, \qquad D' = 0,$$

$$D' = -2\frac{\partial \xi}{\partial v}\frac{\frac{\partial \zeta}{\partial v} - \xi \frac{\partial \eta}{\partial v} + \eta \frac{\partial \xi}{\partial v}}{(\nu_2 + \nu_3 \xi)^2}, \tag{5}$$

$$\mathcal{E} = \frac{4\left(\frac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad \mathcal{F} = 0, \qquad \mathcal{G} = \frac{-4\left(\frac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \tag{6}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the coefficients of the linear element of the spherical representation of S. From these results we have incidentally the well-known fact that the lines of curvature on S are parametric.

However, we are concerned entirely with surfaces whose lines of curvature admit an isothermal spherical representation. From (6) it is seen that a necessary and sufficient condition for this is that *

$$\frac{\partial \xi}{\partial u} - i\frac{\partial \xi}{\partial v} = 0, \tag{7}$$

or by means of (1),

$$\frac{\partial}{\partial u}\log\frac{\nu_3}{\nu_2} + i\frac{\partial}{\partial v}\log\frac{\nu_3}{\nu_2} = 0. \tag{8}$$

It has been assumed that the parameters are isometric, which assumption does not affect the generality of our solution.

If we put

$$u + iv = \alpha, \qquad u - iv = \beta, \tag{9}$$

equation (8) gives the result

$$\nu_3 = \nu_2 A, \tag{10}$$

where A denotes an arbitrary function of α. Since ν_2 and ν_3 satisfy (2), which

* The case where (7) is replaced by $\partial\xi/\partial u + i \cdot \partial\xi/\partial v = 0$ leads to similar results.

by the transformation (9) becomes

$$\frac{\partial^2\theta}{\partial\alpha^2} - \frac{\partial^2\theta}{\partial\beta^2} = -iM\theta, \tag{11}$$

we must have

$$2A'\frac{\partial\nu_2}{\partial\alpha} + \nu_2 A'' = 0,$$

where the accents indicate differentiation with respect to the argument. From this equation we find

$$\nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}}$$

where B denotes an arbitrary function of β, and where the accent denotes differentiation with respect to β. In consequence of (10), we have

$$\nu_1 = \frac{\sqrt{B'}}{\sqrt{A'}}\sigma, \qquad \nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}}, \qquad \nu_3 = \frac{\sqrt{B'}}{\sqrt{A'}}A, \tag{12}$$

where in general σ is a function of α and β, being a solution of

$$\frac{\partial}{\partial\beta}\left[\frac{B'}{A'}\frac{\partial\sigma}{\partial\beta}\right] = \frac{\partial}{\partial\alpha}\left[\frac{B'}{A'}\frac{\partial\sigma}{\partial\alpha}\right], \tag{13}$$

which is obtained by expressing the fact that the functions (12) are solutions of equation (11).

§ 2. *The Determination of* Σ_1 *and of* S_1.

If θ_1 is a particular solution of equation (11), the functions $\bar{\nu}_i$, defined by*

$$\frac{\partial}{\partial\alpha}(\theta_1\bar{\nu}_i) = -\begin{vmatrix}\theta_1 & \nu_i\\ \frac{\partial\theta_1}{\partial\beta} & \frac{\partial\nu_i}{\partial\beta}\end{vmatrix}, \qquad \frac{\partial}{\partial\beta}(\theta_1\bar{\nu}_i) = -\begin{vmatrix}\theta_1 & \nu_i\\ \frac{\partial\theta_1}{\partial\alpha} & \frac{\partial\nu_i}{\partial\alpha}\end{vmatrix}, \tag{14}$$

determine a surface Σ_1, upon which the curves $u = \text{const.}$, $v = \text{const.}$, where u and v are given by (9), are the asymptotic lines; and Σ and Σ_1 are the focal surfaces of a W-congruence, formed by the lines joining corresponding points. Moreover, the coördinates, ξ_1, η_1, ζ_1, of Σ_1 are given by

$$\xi_1 - \xi = \nu_2\bar{\nu}_3 - \nu_3\bar{\nu}_2, \quad \eta_1 - \eta = \nu_3\bar{\nu}_1 - \nu_1\bar{\nu}_3, \quad \zeta_1 - \zeta = \nu_1\bar{\nu}_2 - \nu_2\bar{\nu}_1. \tag{15}$$

In order that Σ_1 may be a surface of the same kind as Σ of § 1, we must have

$$\bar{\nu}_1 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}\sigma_1, \quad \bar{\nu}_2 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}, \quad \bar{\nu}_3 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}A_1, \tag{16}$$

where A_1 and B_1 are functions of α and β alone to be determined and where, in general, σ_1 is a function of both α and β.

* EISENHART, *Differential Geometry*, pp. 417–419.

When the values for $\bar{\nu}_2$ and $\bar{\nu}_3$ are substituted in (14), we obtain four equations which are equivalent to

$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \frac{A_1'}{A-A_1},$$
$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = \frac{(\sqrt{B'})'}{\sqrt{B'}} + \frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\sqrt{A'A_1'}}{A-A_1}, \tag{17}$$

and

$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = \sqrt{A'}\left(\frac{1}{\sqrt{A'}}\right)' + \frac{A'}{A-A_1},$$
$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\sqrt{A'A_1'}}{A-A_1}. \tag{18}$$

The necessary and sufficient condition that these two values of $\partial\theta_1/\partial\alpha$ be equal is readily found to be

$$\sqrt{A'A_1'} = \kappa_1(A-A_1), \tag{19}$$

where κ_1 denotes a constant.

In like manner the condition that the two values of $\partial\theta_1/\partial\beta$ be equivalent is reducible to

$$\sqrt{B'B_1'} = \kappa_1(B-B_1). \tag{20}$$

In consequence of these results, equations (17) and (18) may be written

$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \kappa_1\frac{\sqrt{A_1'}}{\sqrt{A'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = \frac{(\sqrt{B'})'}{\sqrt{B'}} + \kappa_1\frac{\sqrt{B_1'}}{\sqrt{B'}}, \tag{21}$$

and

$$\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = -\frac{(\sqrt{A'})'}{\sqrt{A'}} + \kappa_1\frac{\sqrt{A'}}{\sqrt{A_1'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \kappa_1\frac{\sqrt{B'}}{\sqrt{B_1'}}. \tag{22}$$

When the expression (16) for $\bar{\nu}_1$ is substituted in (14), the resulting equations are reducible by means of (21) and (22) to

$$\frac{\sqrt{A'}}{\sqrt{A_1'}}\frac{\partial\sigma_1}{\partial\alpha} + \kappa_1(\sigma_1-\sigma) + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\partial\sigma}{\partial\beta} = 0,$$
$$\frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\partial\sigma_1}{\partial\beta} + \kappa_1(\sigma_1-\sigma) + \frac{\sqrt{A_1'}}{\sqrt{A'}}\frac{\partial\sigma}{\partial\alpha} = 0. \tag{23}$$

As these equations are linear, the determination of σ_1 requires quadratures only. Hence the problem of the determination of the surfaces Σ_1, when Σ is given in terms of its asymptotic lines, requires the solution of the ordinary differential equation (19) and quadratures.

When these funtions $\sigma_1, \xi_1, \eta_1, \zeta_1$ have been obtained from (23) and (15), we find the coördinates of a surface S_1 by means of equations analogous to (3). If x_1, y_1, z_1 denote the coördinates of S_1 and X_1, Y_1, Z_1 denote the direction cosines of the normal to S_1, we find that

$$\begin{aligned} x_1 + iy_1 - (x + iy) &= R[X_1 + iY_1 - (X + iY)], \\ x_1 - iy_1 - (x - iy) &= R[X_1 - iY_1 - (X - iY)], \\ z_1 - z &= R(Z_1 - Z), \end{aligned} \tag{24}$$

where

$$R = \tfrac{1}{2}\frac{(A\sigma_1 - A_1\sigma)\xi + (A - A_1)\eta + \sigma_1 - \sigma}{A_1 - A}. \tag{25}$$

Hence S and S_1 are the envelope of a two-parameter family of spheres whose radii are defined by (25). The coördinates, x_0, y_0, z_0, of the centers of these spheres are given by

$$\begin{aligned} x_0 + iy_0 &= x + iy - R(X + iY) = \xi\frac{\sigma_1 - \sigma}{A - A_1} - \zeta, \\ x_0 - iy_0 &= x - iy - R(X - iY) = \frac{A_1\sigma - A\sigma_1}{A_1 - A}, \\ 2z_0 &= 2(z - RZ) = \frac{A_1\sigma - A\sigma_1}{A_1 - A}\xi + \eta + \frac{\sigma_1 - \sigma}{A_1 - A}. \end{aligned} \tag{26}$$

We shall refer to the above relation between S and S_1 as a transformation T_{κ_1}, thus indicating the constant κ_1 which appears in the formulas.

§ 3. *Transformation of Minimal Surfaces.*

Minimal surfaces are the only surfaces of negative curvature for which the following conditions are satisfied simultaneously

$$E = G, \qquad \mathcal{E} = \mathcal{G}, \qquad F = 0, \qquad \mathcal{F} = 0.$$

In consequence of (4) and (6), these conditions are equivalent to the following upon Σ:

$$\left(\frac{\partial\xi}{\partial u}\right)^2 + \left(\frac{\partial\xi}{\partial v}\right)^2 = 0,$$

$$\left(\frac{\partial\zeta}{\partial u} - \xi\frac{\partial\eta}{\partial u} + \eta\frac{\partial\xi}{\partial u}\right)^2 + \left(\frac{\partial\zeta}{\partial v} - \xi\frac{\partial\eta}{\partial v} + \eta\frac{\partial\xi}{\partial v}\right)^2 = 0.$$

If we replace the first of these equations by (7), we must replace the second by

$$\frac{\partial\zeta}{\partial\beta} - \xi\frac{\partial\eta}{\partial\beta} + \eta\frac{\partial\xi}{\partial\beta} = 0, \tag{27}$$

if the functions D and D'', given by (5), are to differ in sign.

By means of (1) and (12) equation (27) is reducible to

$$\eta = \frac{1+AB}{A'}\frac{\partial\sigma}{\partial\alpha} - \sigma B. \tag{28}$$

If this equation be differentiated with respect to α and the result equated to the value of $\partial\eta/\partial\alpha$ obtainable from (1) and (12), the resulting equation is reducible by means of (13) to

$$\frac{\partial}{\partial\beta}\log\left(\frac{B'}{A'}\frac{\partial\sigma}{\partial\beta}\right) = \frac{B'A}{1+AB}.$$

From this we get, by integration,

$$\frac{B'}{A'}\frac{\partial\sigma}{\partial\beta} = f(\alpha)(1+AB),$$

where the function f remains to be determined. Hence, integrating again with respect to β, we find

$$\sigma = A'f\int\frac{1+AB}{B'}d\beta + f_1(\alpha). \tag{29}$$

When this value is substituted in (13), we obtain

$$fA - \left(\frac{f_1'}{A'}\right)' = \left[\frac{(A'f)'}{A'}\right]'\int\frac{d\beta}{B'} + \left\{\left[\frac{(A'f)'}{A'}\right]A + A'f\right\}'\int\frac{Bd\beta}{B'}. \tag{30}$$

Differentiating with respect to β, we have

$$\left[\frac{(A'f)'}{A'}\right]' + B\left\{\left[\frac{(A'f)'}{A'}\right]A + A'f\right\}' = 0.$$

Hence the two terms of this equation must be zero, from which we get

$$A'f = c, \tag{31}$$

where c denotes a constant. When this value of f is substituted in (30), we obtain

$$f_1 = c\left[A\int\frac{Ad\alpha}{A'} - \int\frac{A^2d\alpha}{A'}\right] + c_1A + c_2, \tag{32}$$

where c_1 and c_2 denote constants.

From (13) it is seen that if σ is a solution of this equation, so also is $\sigma + c_1A + c_2$ where c_1 and c_2 are arbitrary constants. Hence for the present we shall put $c_1 = c_2 = 0$ and furthermore assume $c = 1$. We shall see presently that this choice of the constants does not restrict the generality of our solution. When these values are substituted in (29), we obtain

$$\sigma = \int\frac{1+AB}{B'}d\beta + A\int\frac{Ad\alpha}{A_1} - \int\frac{A^2d\alpha}{A'}. \tag{33}$$

When this value and (12) are substituted in (1), we find

$$\xi = B,$$

$$\eta = \int \frac{A d\alpha}{A'} + B\int \frac{A^2 d\alpha}{A'} - B\int \frac{d\beta}{B'} + \int \frac{B d\beta}{B'}, \tag{34}$$

$$\zeta = -\int \frac{d\alpha}{A'} - B\int \frac{A d\alpha}{A'} - B\int \frac{B d\beta}{B'} + \int \frac{B^2 d\beta}{B'}.$$

It is readily found that these values satisfy equations (27) and (28).

From (3) we obtain by means of (12) and (34)

$$x = \frac{1}{2}\left(\int \frac{1-A^2}{A'} d\alpha + \int \frac{1-B^2}{B'} d\beta\right),$$

$$y = -\frac{i}{2}\left(\int \frac{1+A^2}{A'} d\alpha - \int \frac{1+B^2}{B'} d\beta\right), \tag{35}$$

$$z = \int \frac{A d\alpha}{A'} + \int \frac{B d\beta}{B'},^*$$

and

$$X = \frac{A+B}{1+AB}, \qquad Y = \frac{i(A-B)}{1+AB}, \qquad Z = \frac{AB-1}{1+AB}. \tag{36}$$

From (35) we find that the linear element of S is

$$ds^2 = \frac{(1+AB)^2}{A'B'} d\alpha d\beta, \tag{37}$$

and from (36) that the linear element of the spherical representation is

$$ds'^2 = \frac{4A'B'}{(1+AB)^2} d\alpha d\beta. \tag{38}$$

The minimal surface $\bar{S}$ adjoint to S is given by equations of the form

$$\frac{\partial \bar{x}}{\partial \alpha} = i\frac{\partial x}{\partial \alpha}, \qquad \frac{\partial \bar{x}}{\partial \beta} = -i\frac{\partial x}{\partial \beta}.$$

Hence

$$\bar{x} = \frac{i}{2}\left(\int \frac{A^2-1}{A'} d\alpha - \int \frac{B^2-1}{B'} d\beta\right),$$

$$\bar{y} = \frac{1}{2}\left(\int \frac{A^2+1}{A'} d\alpha + \int \frac{B^2+1}{B'} d\beta\right), \tag{39}$$

$$\bar{z} = i\int \frac{A d\alpha}{A'} - i\int \frac{B d\beta}{B'}.$$

*Since the equations of any minimal surface can be given this form, the particular choice of the constants in equation (32) did not limit the generality of the discussion.

If we substitute the value (33) for σ and also

$$\sigma_1 = \int \frac{1 + A_1 B_1}{B_1'} d\beta + A_1 \int \frac{A_1 d\alpha}{A_1'} - \int \frac{A_1^2 d\alpha}{A_1'} \tag{40}$$

in equations (23), we find that these equations are identically satisfied in consequence of the relations (19) and (20). Hence the minimal surface S, defined by (35), and the minimal surface S_1, defined by equations of the same form as (35) but with A and B replaced by A_1 and B_1 respectively, constitute the envelope of a two-parameter family of spheres.

From (25) we find that the radius of the sphere is

$$R = \frac{(1 + AB)(1 + A_1 B_1)}{2\sqrt{A'B'} \cdot \sqrt{A_1' B_1'}}. \tag{41}$$

The coördinates of the center of the sphere are readily found by means of (26).

If $\bar{S}_1$ denotes the adjoint of $\bar{S}$, it is defined by equations analogous to (39). From these equations we find that

$$\begin{aligned} \bar{x} - \bar{x}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{AA_1 - 1}{A - A_1} - \frac{BB_1 - 1}{B - B_1}\right), \\ \bar{y} - \bar{y}_1 &= \frac{1}{2\kappa_1^2}\left(\frac{AA_1 + 1}{A - A_1} + \frac{BB_1 + 1}{B - B_1}\right), \\ \bar{z} - \bar{z}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{A + A_1}{A - A_1} - \frac{B + B_1}{B - B_1}\right). \end{aligned} \tag{42}$$

If X_1, Y_1, Z_1 denote the direction-cosines of the parallel normals to S_1 and $\bar{S}_1$, they are given by equations similar to (36). From these values and (42) we find

$$\Sigma X(\bar{x} - \bar{x}_1) = 0, \qquad \Sigma X_1(\bar{x} - \bar{x}_1) = 0.$$

Hence $\bar{S}$ and $\bar{S}_1$ are the focal sheets of the congruence formed by the joins of corresponding points. Moreover, the curves

$$\alpha + \beta = \text{const.}, \qquad \alpha - \beta = \text{const.}$$

on $\bar{S}$ and $\bar{S}_1$ are the asymptotic lines. Hence this congruence is a W-congruence, and the transformation from $\bar{S}$ to $\bar{S}_1$ is the transformation discovered by Thybaut.*

§ 4. *Identification of the Transformation of* § 2 *with a Former Transformation.*

In this section we shall show that the transformation of § 2 is identical with one formerly established in a different manner.†

* Cf. Introduction.

† *Surfaces with isothermal representation, etc.*, these Transactions, vol. 9 (1908), pp. 149–177. A reference to an equation (a) on page b of this memoir will be indicated by [(a) p. b].

Comparing (38) with [(2) p. 151], we have

$$e^{2\theta} = \frac{(1+AB)^2}{4A'B'}.$$

When an equation for S_1 analogous to (38) is compared with [(27) p. 156], we find that

$$\frac{\phi^2}{\omega^2} = \frac{(1+AB)^2(1+A_1B_1)^2}{16A'A_1'B'B_1'}. \tag{43}$$

This identification is permissible in view of the results of § 3 and § 1 of the present and former memoirs respectively.

If we compare (37) with [(1), p. 151], we see that the coördinates of the minimal surface in § 3 are twice as large as in the former memoir. Moreover, the radius of the sphere, as given by [(30′), p. 157], is ϕ/ω, and consequently one half of R as given by (41) and (43). Furthermore, in § 5 of the former memoir it was shown that the case of minimal surfaces in § 1 of that memoir is a particular case of the general transformation of § 4 of the same memoir. Hence, in order to identify the two transformations, it is necessary and sufficient to show that R as given by (25) is equivalent to [cf. (48), p. 161]

$$R = \frac{2}{\omega}\left[m\phi W_1 + \left(\frac{\partial W}{\partial u}\frac{\partial \phi}{\partial u} - \frac{\partial W}{\partial v}\frac{\partial \phi}{\partial v}\right) - W(m\phi - \omega)\right], \tag{44}$$

where m is a constant and W, W_1 measure the distances from the origin to the tangent planes of a surface S and its transform S_1.

From (3) and similar equations for S_1 we have

$$W = -\frac{\eta + A\zeta}{1+AB}, \qquad W_1 = -\frac{\eta_1 + A_1\zeta_1}{1+A_1B_1}.$$

By means of (1), (9) and (12) we find

$$\frac{\partial W}{\partial \alpha} = \frac{A'(B\eta - \zeta)}{(1+AB)^2}, \qquad \frac{\partial W}{\partial \beta} = \frac{AB'(\eta + A\zeta)}{(1+AB)^2} + \frac{\sigma B'}{1+AB}.$$

From [(25), p. 156] and [(6), p. 151], we get (using the notation of the present memoir),

$$\Sigma X_1 \frac{\partial X}{\partial \alpha} = -\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \beta}, \qquad \Sigma X_1 \frac{\partial X}{\partial \beta} = -\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \alpha},$$

which by means of (36) gives

$$\frac{\partial \phi}{\partial \alpha} = -\frac{m\phi[A(A_1B_1 - 1) + A_1 - A^2B_1]}{2A'(1+A_1B_1)},$$

$$\frac{\partial \phi}{\partial \beta} = -\frac{m\phi[B(A_1B_1 - 1) + B_1 - A_1B^2]}{2B'(1+A_1B_1)}.$$

When these values are substituted in (44), we get (25), provided that

$$m = \kappa_1^2.$$

Hence the two transformations are identical.

§ 5. *Theorem of Permutability.*

In this section we establish the following theorem of permutability:

*If a surface S with isothermal representation of its lines of curvature be transformed into two surfaces S_1 and S_2 of the same kind by transformations T_{κ_1} and T_{κ_2} respectively, there exists a surface S' which is the transform of S_1 and of S_2 by transformations T'_{κ_2} and T'_{κ_1} respectively; and this surface S' can be obtained without quadratures.**

In order that there may be such a transform S', there must exist functions $A^{(1)}$ and $B^{(1)}$ satisfying the following equations analogous to (19) and (20),

$$\begin{aligned} &\sqrt{A'_1 A^{(1)'}} = \kappa'_1 (A_1 - A^{(1)}), \qquad \sqrt{B'_1 B^{(1)'}} = \kappa'_1 (B_1 - B^{(1)}), \\ &\sqrt{A'_2 A^{(1)'}} = \kappa'_2 (A_2 - A^{(1)}), \qquad \sqrt{B'_2 B^{(1)'}} = \kappa'_2 (B_2 - B^{(1)}), \end{aligned} \tag{45}$$

where for the present the constants κ'_1 and κ'_2 are undetermined. Also there must exist a function σ' satisfying equations analogous to (23), which by means of the latter can be given the form

$$\begin{aligned} &\frac{\sqrt{A'_1}}{\sqrt{A^{(1)'}}} \frac{\partial \sigma'}{\partial \alpha} + \kappa'_1 (\sigma' - \sigma_1) + \frac{\sqrt{B'}}{\sqrt{B^{(1)'}}} \left[\kappa_1 (\sigma - \sigma_1) - \frac{\sqrt{A'_1}}{\sqrt{A'}} \frac{\partial \sigma}{\partial \alpha} \right] = 0, \\ &\frac{\sqrt{B^{(1)'}}}{\sqrt{B'_1}} \frac{\partial \sigma'}{\partial \beta} + \kappa'_1 (\sigma' - \sigma_1) + \frac{\sqrt{A^{(1)'}}}{\sqrt{A'}} \left[\kappa_1 (\sigma - \sigma_1) - \frac{\sqrt{B'}}{\sqrt{B'_1}} \frac{\partial \sigma}{\partial \beta} \right] = 0, \end{aligned} \tag{46}$$

and similar equations obtained by changing the subscripts 1 into 2.

Eliminating $\partial \sigma'/\partial \alpha$ and $\partial \sigma'/\partial \beta$ from these four equations, we obtain

$$\begin{aligned} &(\kappa'_1 \sqrt{B'_1} - \kappa'_2 \sqrt{B'_2}) \sigma' - (\kappa'_1 \sqrt{B'_1} \sigma_1 - \kappa'_2 \sqrt{B'_2} \sigma_2) \\ &\qquad + \frac{\sqrt{A^{(1)'}}}{\sqrt{A'}} [\kappa_1 \sqrt{B'_1} (\sigma - \sigma_1) - \kappa_2 \sqrt{B'_2} (\sigma - \sigma_2)] = 0, \\ &(\kappa'_1 \sqrt{A'_2} - \kappa'_2 \sqrt{A'_1}) \sigma' - (\kappa'_1 \sqrt{A'_2} \sigma_1 - \kappa'_2 \sqrt{A'_1} \sigma_2) \\ &\qquad + \frac{\sqrt{B'}}{\sqrt{B^{(1)'}}} [\kappa_1 \sqrt{A'_2} (\sigma - \sigma_1) - \kappa_2 \sqrt{A'_1} (\sigma - \sigma_2)] = 0. \end{aligned} \tag{47}$$

* Cf. former memoir, l. c., p. 166.

By means of (19) and (20) equations (45) are reducible to

$$\sqrt{A^{(1)'}} = \frac{\kappa_1'\kappa_2'}{\kappa_1\kappa_2}\sqrt{A'}\frac{\kappa_1\sqrt{A_2'} - \kappa_2\sqrt{A_1'}}{\kappa_2'\sqrt{A_1'} - \kappa_1'\sqrt{A_2'}}, \tag{48}$$

and the same equation in the B's. When these values are substituted in (47), it is found that the two equations are consistent if and only if

$$\kappa_1' = \pm\kappa_1, \qquad \kappa_2' = \pm\kappa_2 \tag{49}$$

or

$$\kappa_1' = \pm\kappa_2, \qquad \kappa_2' = \pm\kappa_1, \tag{50}$$

where the same sign must be used in every case.

When the values (49) are used we get

$$\sqrt{A^{(1)'}} = \mp\sqrt{A'}, \qquad \sigma' = \sigma,$$

so that the case (49) is to be excluded.

For the values (50) equation (48) is reducible to

$$A^{(1)} = \frac{\kappa_1^2 A_2(A - A_1) - \kappa_2^2 A_1(A - A_2)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}, \tag{51}$$

and the two expressions (47) may be given the form

$$\begin{aligned}\sigma' = {} & \frac{\kappa_1^2(B - B_1) - \kappa_2^2(B - B_2)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}\frac{A_1 - A_2}{B_1 - B_2}\sigma \\ & + \frac{\kappa_1^2 - \kappa_2^2}{B_1 - B_2}\frac{\sigma_2(A - A_1)(B - B_2) - \sigma_1(A - A_2)(B - B_1)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}.\end{aligned} \tag{52}$$

Hence the theorem is established if these values satisfy (46), as they can be shown to do.

PRINCETON,
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ON THE BASE OF A RELATIVE NUMBER-FIELD, WITH AN APPLICATION TO THE COMPOSITION OF FIELDS*

BY

G. E. WAHLIN

It is a well known fact that in an algebraic number-field of degree n there exist n integers $\alpha_1, \cdots, \alpha_n$ such that every integer of the field can be expressed in the form

$$\omega = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n,$$

where $x_1, x_2, x_3, \cdots, x_n$ are rational integers.

If we now consider a field relative to a given subfield of this field, from the nature of the proof of the above fact it is evident that, if the subfield in question is such that the number of classes of ideals in this field is one, and r the degree of the larger field relative to its subfield, then there exist r integers $\beta_1, \beta_2, \cdots, \beta_r$ such that every integer in the larger field can be expressed in the form

$$\omega = y_1\beta_1 + y_2\beta_2 + \cdots + y_r\beta_r,$$

where now $y_1, y_2, \cdots, y_r$ are integers in the subfield.

When, however, the number of classes of the subfield is greater than one, this is not the case. SOMMER† has shown that in this case for a field which is of the second degree relative to a subfield of the second degree the four numbers composing the base may be taken to be $\omega_1, \omega_2, \beta_1\Omega, \beta_2\Omega$, where ω_1, ω_2 is the base of the subfield and β_1, β_2 the base of an ideal in the subfield and Ω a number of the larger field, not necessarily an integer, but such that $\beta_1\Omega$ and $\beta_2\Omega$ are integers in this field.

In the first part of this paper I establish a similar form for the base of any algebraic number-field relative to any subfield, and in the last part I apply this to the study of the discriminant of the field.

The Base of a Relative Field.

Let K be an algebraic number-field of degree N which contains the subfield k of degree n. We then know that $r = N/n$ is the degree of K relative to k, and if ϑ be any integer which determines the field K, then ϑ is the root of an

* Presented to the Society (Chicago), December 31, 1909.

† SOMMER, *Vorlesungen über Zahlentheorie*, p. 299.

equation

$$x^r + \alpha_1 x^{r-1} + \alpha_2 x^{r-2} + \cdots + \alpha_r = 0, \tag{1}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_r$ are integers in k. We therefore know that every number of K can be expressed in the form

$$\omega = \rho_1 + \rho_2 \vartheta + \rho_3 \vartheta^2 + \cdots + \rho_r \vartheta^{r-1}, \tag{2}$$

where $\rho_1, \rho_2, \cdots, \rho_r$ are numbers of the subfield k. Then, in the same manner as when we consider a field relative to the domain of rational integers, we can show that every integer in K can be expressed in the form

$$\omega = \frac{A_1 + A_2 \vartheta + A_3 \vartheta^2 + \cdots + A_r \vartheta^{r-1}}{D_k(\vartheta)}, \tag{3}$$

where $A_1, A_2, \cdots, A_r$ are integers in k and $D_k(\vartheta)$ is the relative discriminant of ϑ with respect to k.*

Let us now consider all integers of K for which $A_{\tau+1} = A_{\tau+2} = \cdots = A_r = 0$, where τ is one of the integers $1, 2, 3, \cdots, r$. These integers we can indicate as follows

$$\omega_\tau^{(i)} = \frac{O_{1\tau}^{(i)} + O_{2\tau}^{(i)} \vartheta + O_{3\tau}^{(i)} \vartheta^2 + \cdots + O_{\tau\tau}^{(i)} \vartheta^{\tau-1}}{D_k(\vartheta)} \quad (i = 1, 2, 3, \cdots). \tag{4}$$

It is evident that any integer of the type (4) when multiplied by an integer of k gives another integer of the same type, and the sum of any number of such integers is also an integer of this type. But this means that the product of any one of the coefficients $O_{\tau\tau}^{(i)}$ by an integer of k gives another of these integers and, moreover, the sum of any number of them is also another integer of the same set. Hence, since all the integers $O_{\tau\tau}^{(i)}$ belong to k, they form an ideal in this field, which ideal is the highest common ideal factor of all these coefficients. This ideal we shall designate by O_τ.

Thus, for $\tau = 1, 2, 3, \cdots, r$ we get r ideals $O_1, O_2, O_3, \cdots, O_r$ in k. But since $\omega_1^{(i)} = O_{11}^{(i)} / D_k(\vartheta)$ is a number of k and, moreover, from our assumption an integer, and since evidently all integers of k are thus represented, k being a subfield of K, and r being the lowest degree of the equation which ϑ satisfies in k, we conclude that $O_{11}^{(i)}$ $(i = 1, 2, 3, \cdots)$ represent all multiples $D_k(\vartheta)$ in k and therefore the ideal O_1 is the principal ideal $[D_k(\vartheta)]$. Moreover, since $\vartheta \cdot \omega_{\tau-1}^{(i)}$ evidently is an integer of K of the type (4), we conclude that all the integers of $O_{\tau-1}$ also belong to O_τ, and hence that $O_{\tau-1}$ is divisible by O_τ. Therefore, in the sequence $O_1, O_2, O_3, \cdots, O_r$ each ideal is a factor of all that precede it in the sequence, and hence all of them are factors of $D_k(\vartheta)$. Therefore, the integer $D_k(\vartheta)$ belongs to each one of these ideals. But we know that if we choose arbitrarily from an ideal any integer

* HILBERT, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 4 (1894–95), p. 180.

there exists another integer such that the given ideal is the greatest common divisor of these two integers. Hence each of the above ideals must contain an integer such that the ideal in question is the greatest common divisor of this integer and $D_k(\vartheta)$. If we then let this integer in O_τ be the coefficient $O_{\tau\tau}^{(1)}$ we can write

$$O_\tau = [O_{\tau\tau}^{(1)},\ D_k(\vartheta)] \qquad (\tau = 1, 2, 3, \cdots, r).$$

Let us now put $D_k(\vartheta)/O_\tau = I_\tau$. Evidently I_τ is relatively prime to $O_{\tau\tau}^{(1)}$ and, hence, there exists in k an integer β such that

$$O_{\tau\tau}^{(1)} \cdot \beta \equiv 1 \qquad (I_\tau).$$

Hence, if we indicate by $\bar{\omega}_\tau$ the numerator in the expression for $\omega_\tau^{(1)}$, we have the following congruence

$$\bar{\omega}_\tau \cdot \beta = (O_{1\tau}^{(1)} + O_{2\tau}^{(1)}\vartheta + O_{3\tau}^{(1)}\vartheta^2 + \cdots + O_{\tau\tau}^{(1)}\vartheta^{\tau-1}) \cdot \beta \equiv \bar{\Omega}_\tau \qquad (I_\tau),$$

where

$$\bar{\Omega}_\tau = A_{1\tau} + A_{2\tau}\vartheta + A_{3\tau}\vartheta^2 + \cdots + A_{\tau-1\tau}\vartheta^{\tau-2} + \vartheta^{\tau-1}.$$

We observe that in $\bar{\Omega}_\tau$ the coefficient of $\vartheta^{\tau-1}$ is unity. But since $\omega_\tau^{(1)} = \bar{\omega}_\tau / D_k(\vartheta)$ is an integer, it follows that $\bar{\omega}_\tau$ is divisible by $D_k(\vartheta)$ and hence by I_τ. From the last congruence it follows that also $\bar{\Omega}_\tau$ is divisible by I_τ.

The quotient $\Omega_\tau = \bar{\Omega}_\tau / D_k(\vartheta)$ is not necessarily an integer in K, but $D_k(\vartheta) \cdot \Omega_\tau$ and $O_{\tau\tau}^{(1)}\Omega_\tau$ are evidently both integers in K.

Let

$$\omega = \frac{\beta_1 + \beta_2\vartheta + \beta_3\vartheta^2 + \cdots + \beta_r\vartheta^{r-1}}{D_k(\vartheta)} \tag{5}$$

be any integer in K. From what we have seen above β_r belongs to the ideal $O_r = [O_{rr}^{(1)},\ D_k(\vartheta)]$ and, hence, there exist in k two integers λ_r and μ_r such that

$$\lambda_r O_{rr}^{(1)} + \mu_r D_k(\vartheta) = \beta_r. \tag{6}$$

But then

$$\beta_r \Omega_r = \lambda_r O_{rr}^{(1)} \Omega_r + \mu_r D_k(\vartheta) \cdot \Omega_r,$$

which is evidently an integer in K and, moreover, this integer when expressed in the form (4) has the same coefficient of ϑ^{r-1} as ω in (5). Hence $\omega - \beta_r\Omega_r$ is an integer of the type

$$\frac{\beta_1' + \beta_2'\vartheta + \beta_3'\vartheta^2 + \cdots \beta_{r-1}'\vartheta^{r-2}}{D_k(\vartheta)}.$$

In the same manner as above we can determine λ_{r-1} and μ_{r-1} such that

$$\beta_{r-1}' = \lambda_{r-1} O_{r-1\,r-1}^{(1)} + \mu_r D_k(\vartheta), \tag{6^a}$$

and $\beta_{r-1}'\Omega_{r-1}$ is then an integer of K of the type (4) when $\tau = r-1$ and the coefficient of ϑ^{r-2} is β_{r-1}'.

Hence we have

$$\omega - \beta_r \Omega_r - \beta'_{r-1}\Omega_{r-1} = \frac{\beta''_1 + \beta''_2 \vartheta + \cdots + \beta''_{r-2}\vartheta^{r-3}}{D_k(\vartheta)},$$

which is again an integer.

By continuing in this manner we evidently shall at last have

$$\omega - \beta_r \Omega_r - \beta'_{r-2}\Omega_{r-1} - \cdots - \beta_1^{(r-1)}\Omega_1 = 0,$$

whence

$$\omega = \beta_1^{(r-1)}\Omega_1 + \beta_2^{(r-2)}\Omega_2 + \cdots + \beta_r \Omega_r, \tag{7}$$

where, for all values of s from 1 to r, $\beta_s^{(r-s)}$ is an integer belonging to the ideal O_s. Now ω is any integer of K and, hence, every integer of K can be represented in the form (7). Moreover, since $O_{\tau\tau}^{(1)}\Omega_\tau$ and $D_k(\vartheta)\Omega_\tau$ are both integers in K, from (6) and (6^a) and the similar expressions for the remaining $\beta_s^{(r-s)}$ we can conclude that when $\beta_s^{(r-s)}$ belongs to the ideal O_s all numbers represented by (7) are integers of the field K.

If we now let $(i_{1\tau}, i_{2\tau}, \cdots, i_{n\tau})$ be the base of the ideal O_τ, then evidently the N integers $i_{\lambda\tau}\Omega_\tau (\lambda = 1, 2, 3 \cdots n;\ \tau = 1, 2, 3 \cdots r)$ form a base of all the integers in the field K. By replacing in this base $\Omega_1, \Omega_2, \Omega_3, \cdots, \Omega_r$ by their relative conjugates with respect to the field k we get the basis of the respective relative conjugate fields.

The Relative Discriminant of K.

Let us now apply this result to the relative discriminant of the field K with respect to k. The relative discriminant is defined as the greatest common factor of the squares of all the determinants formed from the matrix of the base of K and its relative conjugates. It is easily seen that each of the determinants can be written

$$i_{\lambda_1\tau_1} \cdot i_{\lambda_2\tau_2} \cdots i_{\lambda_r\tau_r} \begin{vmatrix} \Omega_{\tau_1}, & \Omega_{\tau_2}, & \cdots, & \Omega_{\tau_r} \\ \Omega_{\tau_1}^{(1)}, & \Omega_{\tau_2}^{(1)}, & \cdots, & \Omega_{\tau_r}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ \Omega_{\tau_1}^{(r-1)}, & \Omega_{\tau_2}^{(r-1)}, & \cdots, & \Omega_{\tau_r}^{(r-1)} \end{vmatrix}, \tag{8}$$

where the upper index is used to designate the relative conjugates. When two of the subscripts τ are alike the expression (8) vanishes. Where no two are alike the determinant factor in (8) is, aside from sign, the same for all determinants of the matrix, and hence the determinant

$$\left|\Omega_{\tau_k}^{(i)}\right|$$

must be a common factor of all the determinants of the matrix. Moreover, all the numbers of the ideal product $O_1 \cdot O_2 \cdot O_3 \cdots O_r$ can be represented linearly by means of the products

$$\pi = i_{\lambda_1\tau_1} \cdot i_{\lambda_2\tau_2} \cdots i_{\lambda_r\tau_r}$$

and hence the greatest common factor of the products π must be a factor of the product $O_1 \cdot O_2 \cdots O_r$. But each π is divisible by $O_1 \cdot O_2 \cdots O_r$, there being in π one number from each of these ideals, and hence their highest common factor must be divisible by $O_1 \cdot O_2 \cdots O_r$. Thus $O_1 \cdot O_2 \cdot O_2 \cdots O_r$ is itself the highest common factor of the products π, and hence the relative discriminant of K is

$$D_k = O_1^2 \cdot O_2^2 \cdot O_3^2 \cdots O_r^2 \, |\, \Omega_{\tau_k}^{(i)} |^2,$$

which is an ideal in the subfield k. Since $\Omega_\tau^{(i)} = \bar{\Omega}_\tau^{(i)} / D_k(\vartheta)$ we can write

$$D_k = \frac{O_2^2 \cdot O_3^2 \cdots O_r^2}{D_k(\vartheta)^{2r-2}} \, |\, \bar{\Omega}_\tau^{(i)} |^2$$

remembering that O_1 is the principal ideal $[D_k(\vartheta)]$. Again since

$$\bar{\Omega}_\tau^{(i)} = A_1 + A_2 \vartheta^{(i)} + \cdots + \vartheta^{(i)^{r-1}},$$

we have

$$D_k = \frac{O_2^2 \cdot O_3^2 \cdots O_r^2}{D_k(\vartheta)^{2r-2}} \, D_k(\vartheta).$$

If as before we put $D_k(\vartheta) / O_r = I_r$, we have

$$D_k = \frac{D_k(\vartheta)}{I_2^2 \cdot I_3^2 \cdots I_r^2}.$$

It is easily seen from the previous deduction that the ideals $I_1, I_2, \cdots, I_r$ are those factors of the relative discriminant of ϑ for which as moduli ϑ is the root of a congruence of lower degree than r in k.

The Discriminant of a Field Formed by the Composition of Two Fields.

We shall next make an application of the relative discriminant of a field in the form found above to determine the nature of the discriminant of a field formed by the composition of two fields, in the case where the degree of the compounded field is equal to the product of the degrees of the two fields from which it is formed, divided by the degree of their greatest common subfield.

The problem of the composition of fields has been studied by HENSEL,* but his results are of a different nature from those obtained by the method here used.

I shall suppose that the field K of degree N is formed by composition from the two fields k_1 and k_2, of degrees n_1 and n_2 respectively. Moreover, let κ be the greatest common subfield to k_1 and k_2 and let the degree of κ be ν. If we let P_1 and P_2 be the degrees of k_1 and k_2 relative to κ, and r_1 and r_2 the degrees of K relative to k_1 and k_2, respectively, we have

$$n_1 = \nu P_1, \qquad n_2 = \nu P_2, \qquad N = r_1 n_1 = r_2 n_2.$$

* HENSEL: *Journal für die reine und angewandte Mathematik*, vol. 105 (1889), p. 329; vol. 120 (1899), p. 99.

But by hypothesis $N = n_1 \cdot n_2/\nu = P_2 n_1$. From this and $N = r_1 n_1$ we conclude that $r_1 = P_2$, and in the same way $r_2 = P_1$. Therefore $N = r_1 \cdot r_2 \cdot \nu$.

I shall indicate by D, d_1, d_2, δ respectively the discriminants of K, k_1, k_2, κ, and the relative discriminant by using as subscript the symbol of the field relative to which it is taken. I shall, moreover, indicate by $N_1(\ )$, $N_2(\ )$, $N(\ )$ the norms taken respectively in the fields k_1, k_2, κ and by N_{k_1}, N_{k_2}, N_κ the relative norms.

From what we have seen above we know that $r_1 \cdot r_2$ is the relative degree of K with respect to κ.

If now α_2 is a number which determines k_2, then α_2 will determine K relative to k_1, and we can therefore write

$$D_{k_1} = \frac{O_{12}^2 \cdot O_{13}^2 \cdots O_{1r_1}^2}{D_{k_1}(\alpha_2)^{2r_1-2}} D_{k_1}(\alpha_2), \tag{9}$$

$$d_{2\kappa} = \frac{O_2^2 \cdot O_3^2 \cdots O_{r_1}^2}{D_\kappa(\alpha_2)^{2r_1-2}} D_\kappa(\alpha_2), \tag{10}$$

where $d_{2\kappa}$ is the relative discriminant of k_2 with respect to κ. But since $P_2 = r_1$ we have

$$D_\kappa(\alpha_2) = D_{k_1}(\alpha_2),$$

because the equation in k_1, which α_2 satisfies, is then the same as the equation in κ, the subfield of k_1. Now $O_{1\tau}$ is an ideal in k_1 and O_τ is an ideal in κ, and since κ is a subfield of k_1 from the definition of $O_{1\tau}$ and O_τ, it follows that O_τ is contained in $O_{1\tau}$ and, hence, $O_{1\tau}$ is a factor of O_τ. Hence $D_\kappa(\alpha_2)/O_\tau = I_\tau$ is a factor of $D_\kappa(\alpha_2)/O_{1\tau} = I_{1\tau}$. If we therefore put

$$\frac{I_{12} \cdot I_{13} \cdots I_{1r_1}}{I_2 \cdot I_3 \cdots I_{r_1}} = T,$$

we see that T contains all the ideal factors of $D_\kappa(\alpha_2)$ in k_1 for which α_2 will satisfy a congruence of degree less than r_1 in k_1 but not in the κ.

Hence from (9) and (10)

$$d_{2\kappa} = T^2 D_{k_1}.$$

Taking the norm in k_1, we get

$$N_1(d_{2\kappa}) = N_1(T^2) N_1(D_{k_1}),$$

or since $d_{2\kappa}$ is an ideal in κ,

$$N(d_{2\kappa}^{r_2}) = N_1(T^2) \cdot N_1(D_{k_1}).$$

But then from the known relation

$$d_2 = \delta^{r_1} N(d_{2\kappa}) \tag{11}$$

* BACHMANN, *Allgemeine Arithmetik der Zahlenkörper*, p. 452.

we have

$$\frac{d_2^{r_2}}{\delta^{r_1 r_2}} = N_1(T^2) \cdot N_1(D_{k_1}). \tag{12}$$

Similar to (11) we have (Bachmann, loc. cit.)

$$D = d_1^{r_1} \cdot N_1(D_{k_1}),$$

and from this we finally have

$$D = \frac{d_1^{r_1} \cdot d_2^{r_2}}{\delta^{r_1 r_2} \cdot N_1(T^2)}. \tag{13}$$

From (12) we see that every prime factor of $N_1(T^2)$ must be a factor of d_2, and if in the previous work we interchange α_1 and α_2, and k_1 and k_2 we should get another ideal T' and $N_2(T'^2)$ would be such that every prime factor of this would be a factor of d_1. But as in either case the final result (13) would have to be the same, we see that

$$N_1(T^2) = N_2(T'^2),$$

and this therefore contains only such prime factors as are common to d_1 and d_2.

URBANA, ILL.

THE STRAIN OF A NON-GRAVITATING SPHERE OF VARIABLE DENSITY*

BY

L. M. HOSKINS

§ 1. *Introductory.*

The mathematical problem of the strain of an elastic sphere is of special physical interest because of its relation to the question of the elastic yielding of the earth to disturbing forces. The cases of the problem which have been solved, however, involve assumptions which depart from what is known or thought probable in the case of the earth. The solutions of LAMÉ and KELVIN apply to a homogeneous body acted upon by bodily forces which are known functions of the coördinates. This case disagrees with the facts as regards the earth in two particulars: (1) The earth is certainly far from homogeneous in density and is very likely far from homogeneous in elastic properties. (2) The important bodily forces of self-gravitation cannot be expressed as functions of the coördinates in advance of the solution, since the changes in these forces caused by the strain are of the same order of importance as the external disturbing forces; it is only by assuming incompressibility that it becomes possible to express the gravitational forces without the use of unknown functions.

In a previous paper† was given a solution of the problem for a compressible sphere initially of uniform density, taking account of the changes in the gravitational forces caused by the strain.

In the present paper will be found a solution for the case of a non-gravitating sphere of which the density is a given function of the distance from the center, the elastic moduli being assumed to be uniform in value throughout the body.

By combining these two cases, and introducing the additional assumption that the elastic moduli are functions of the distance from the center, we should have perhaps the nearest possible approximation to a completely general statement of the problem presented by the actual earth. While there is no difficulty in forming the differential equations for this general problem in a manner analogous to that employed in the two cases which have been solved, a solution of these general equations has not thus far been obtained.

* Presented to the Society, San Francisco Section, February 26, 1910.

† Transactions of the American Mathematical Society, vol. 11 (1910), p. 203.

The case treated in this paper appears, however, to possess some interest and importance, being a generalization in one important particular of the problem which formed the basis of the original estimates of the rigidity of the earth.

§ 2. *Statement of Problem.*

The problem here treated is the following:

To determine the strain of an elastic sphere of which the density is a given function of the distance from the center, under the action of bodily forces having a potential, but free from stress on the bounding surface. The potential may be any function which, when expressed in polar coördinates, is developable as to θ and ϕ in a series of spherical surface harmonics.

§ 3. *Equations of Equilibrium of Volume-Element.*

Using polar coördinates r, θ, ϕ, let u_r, u_θ, u_ϕ be the elastic displacements, ϖ_r, ϖ_θ, ϖ_ϕ the rotation-components, Δ the cubical dilatation, F_r, F_θ, F_ϕ the components of bodily force per unit mass, $\lambda + \frac{2}{3}\mu$, μ the bulk-modulus and rigidity-modulus, ρ the density.

The three equations of equilibrium for an element of volume are the following:

$$(1)\qquad (\lambda+2\mu)\frac{\partial\Delta}{\partial r} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\varpi_\phi \sin\theta) - \frac{\partial\varpi_\theta}{\partial\phi}\right] + \rho F_r = 0,$$

$$(2)\qquad (\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial\varpi_r}{\partial\phi} - \frac{\partial}{\partial r}(r\varpi_\phi \sin\theta)\right] + \rho F_\theta = 0,$$

$$(3)\qquad (\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial\Delta}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\varpi_\theta) - \frac{\partial\varpi_r}{\partial\theta}\right] + \rho F_\phi = 0;$$

in which

$$(4)\qquad \Delta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r \sin\theta) + \frac{\partial}{\partial\theta}(ru_\theta \sin\theta) + \frac{\partial}{\partial\phi}(ru_\phi)\right],$$

$$(5)\qquad 2\varpi_r = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(ru_\phi \sin\theta) - \frac{\partial}{\partial\phi}(ru_\theta)\right],$$

$$(6)\qquad 2\varpi_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi \sin\theta)\right],$$

$$(7)\qquad 2\varpi_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Let W denote the potential of the bodily forces, and assume that

$$(8)\qquad W = \Sigma R_i S_i,$$

in which S_i denotes a spherical surface harmonic of order i, and R_i a function of r. Then

$$F_r = \Sigma \frac{dR_i}{dr} S_i,$$

$$F_\theta = \Sigma \frac{R_i}{r} \frac{\partial S_i}{\partial \theta}, \tag{9}$$

$$F_\phi = \Sigma \frac{R_i}{r \sin \theta} \frac{\partial S_i}{\partial \phi}.$$

A solution of equations (1), (2) and (3) may be formed by putting together solutions corresponding to the separate terms in the value of W. We therefore confine the following discussion to the solution corresponding to a single term $R_i S_i$.

We now assume

$$u_r = u S_i, \qquad u_\theta = v \frac{\partial S_i}{\partial \theta}, \qquad u_\phi = \frac{v}{\sin \theta} \frac{\partial S_i}{\partial \phi}, \tag{10}$$

in which u and v are functions of r only. Substituting these values in equations (4), $\cdots$, (7), and making use of the differential equation

$$\frac{1}{\sin^2 \theta} \frac{\partial^2 S_i}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial S_i}{\partial \theta} \right) + i(i+1) S_i = 0 \tag{11}$$

which is satisfied by any spherical surface harmonic of order i, we find

$$\Delta = \left[\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1) \frac{v}{r} \right] S_i, \tag{12}$$

$$2\varpi_r = 0, \tag{13}$$

$$2\varpi_\theta = \frac{1}{r} \left[u - \frac{d(rv)}{dr} \right] \frac{1}{\sin \theta} \frac{\partial S_i}{\partial \phi}, \tag{14}$$

$$2\varpi_\phi = -\frac{1}{r} \left[u - \frac{d(rv)}{dr} \right] \frac{\partial S_i}{\partial \theta}. \tag{15}$$

If the foregoing values of F_r, F_θ, F_ϕ, Δ, ϖ_r, ϖ_θ and ϖ_ϕ be substituted in (1), (2) and (3), these are freed from θ and ϕ. Equations (2) and (3) in fact become identical, and there remain to be satisfied two ordinary differential equations for determining u, v as functions of r. Writing R for R_i, these equations are the following:

$$(\lambda + 2\mu) \frac{d}{dr} \left(\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1) \frac{v}{r} \right) + \frac{i(i+1)\mu}{r^2} \left(\frac{d(rv)}{dr} - u \right) + \rho \frac{dR}{dr} = 0, \tag{16}$$

$$(17)\quad (\lambda+2\mu)\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r}\right)+\mu\frac{d}{dr}\left(\frac{d(rv)}{dr}-u\right)+\rho R=0.$$

It is now convenient to replace r by $x=r/a$ as independent variable, and to introduce y and z, defined by the equations

$$(18)\qquad y=\frac{\Delta}{S_i}=\frac{1}{r^2}\frac{d(r^2u)}{dr}-i(i+1)\frac{v}{r},$$

$$(19)\qquad z=\frac{2\varpi_\phi}{\partial S_i/\partial\theta}=\frac{1}{r}\left(\frac{d(rv)}{dr}-u\right),$$

thus reducing the differential equations to the following:

$$(20)\qquad (\lambda+2\mu)\frac{dy}{dx}+i(i+1)\mu\frac{z}{x}+\rho\frac{dR}{dx}=0,$$

$$(21)\qquad (\lambda+2\mu)y+\mu\frac{d(xz)}{dx}+\rho R=0.$$

§ 4. *Surface Conditions.*

We now have to express the condition that the bounding surface is free from stress, that is,

$$(22)\qquad \widehat{rr}=0,\qquad \widehat{r\theta}=0,\qquad \widehat{r\phi}=0 \qquad (x=1).$$

Using (10) in the general formulas for these stresses, we find

$$(23)\qquad \widehat{rr}=\lambda\Delta+2\mu\frac{\partial u_r}{\partial r}=\left(\lambda y+2\mu\frac{du}{dr}\right)S_i,$$

$$(24)\qquad \widehat{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)=\mu\left(\frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}\right)\frac{\partial S_i}{\partial\theta},$$

$$(25)\qquad \widehat{r\phi}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)=\mu\left(\frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi},$$

so that equations (22) reduce to the two

$$(26)\qquad \lambda y+2\mu\frac{du}{dr}=0,$$

$$(27)\qquad \frac{dv}{dr}-\frac{v}{r}+\frac{u}{r}=0,$$

to be satisfied when $x=1$.

§ 5. *Solution of Differential Equations.*

By eliminating successively z and y between equations (20) and (21) we obtain the two equations

$$(28)\quad \left(x^2\frac{d^2}{dx^2}+2x\frac{d}{dx}-i(i+1)\right)\left((\lambda+2\mu)y+\rho R\right)-\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)=0,$$

$$(29)\quad \mu\left(x^2\frac{d^2z}{dx^2}+2x\frac{dz}{dx}-i(i+1)z\right)+xR\frac{d\rho}{dx}=0.$$

The solutions of these equations are the following:

$$(30)\quad \begin{aligned}(2i+1)[(\lambda+2\mu)y+\rho R]&=x^i\int x^{-i-1}\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)dx\\&\quad-x^{-i-1}\int x^i\frac{d}{dx}\left(x^2R\frac{d\rho}{dx}\right)dx,\end{aligned}$$

$$(31)\quad (2i+1)\mu z=-x^i\int x^{-i-1}\left(xR\frac{d\rho}{dx}\right)dx+x^{-i-1}\int x^i\left(xR\frac{d\rho}{dx}\right)dx,$$

in which, for the general solutions, each integral must be understood to involve an arbitrary constant. These constants are here omitted, since they are accounted for in the complementary solutions to be obtained presently. Modifying by integration by parts, we find

$$(32)\quad (\lambda+2\mu)y=-\frac{1}{2i+1}\left[(i+1)x^i\int\rho\frac{d}{dx}\left(\frac{R}{x^i}\right)dx+\frac{i}{x^{i+1}}\int\rho\frac{d(Rx^{i+1})}{dx}dx\right],$$

$$(33)\quad \mu z=\frac{1}{2i+1}\left[x^i\int\rho\frac{d}{dx}\left(\frac{R}{x^i}\right)dx-\frac{1}{x^{i+1}}\int\rho\frac{d(Rx^{i+1})}{dx}dx\right].$$

These values of y and z may now be substituted in (18) and (19), giving two simultaneous linear differential equations for determining u and v. The solutions of these equations may conveniently be expressed as follows:

From (18) and (19) we readily obtain the equations

$$(34)\quad \frac{d}{dx}(iv-u)+\frac{i+2}{x}(iv-u)=a(iz-y),$$

$$(35)\quad \frac{d}{dx}[(i+1)v+u]-\frac{i-1}{x}[(i+1)v+u]=a[(i+1)z+y],$$

of which the solutions are

$$(36)\quad iv-u=\frac{a}{x^{i+2}}\int(iz-y)x^{i+2}dx,$$

$$(37)\qquad (i+1)v+u=ax^{i-1}\int\frac{(i+1)z+y}{x^{i-1}}\,dx.$$

These are easily solved for u and v.

The forms of the complementary solutions are easily seen by considering the constants of integration in (32), (33), (36) and (37). The values of the constants must be determined so as to satisfy (18), (19), (20) and (21). The complementary solutions are thus found to be the following:

$$\begin{aligned}
y&=C_ix^i+C_{-i-1}x^{-i-1},\\
z&=D_ix^i+D_{-i-1}x^{-i-1},\\
(38)\qquad \frac{u}{r}&=A_ix^i+A_{-i-1}x^{-i-1}+A_{i-2}x^{i-2}+A_{-i-3}x^{-i-3},\\
\frac{v}{r}&=B_ix^i+B_{-i-1}x^{-i-1}+B_{i-2}x^{i-2}+B_{-i-3}x^{-i-3};
\end{aligned}$$

in which the twelve constants must satisfy the eight equations

$$\begin{aligned}
(\lambda+2\mu)C_i+(i+1)\mu D_i&=0,\\
(i+3)A_i-i(i+1)B_i-C_i&=0,\\
A_i-(i+2)B_i+D_i&=0,\\
(\lambda+2\mu)C_{-i-1}-i\mu D_{-i-1}&=0,\\
(i-2)A_{-i-1}+i(i+1)B_{-i-1}+C_{-i-1}&=0,\\
A_{-i-1}+(i-1)B_{-i-1}+D_{-i-1}&=0,\\
A_{i-2}-iB_{i-2}&=0,\\
A_{-i-3}+(i+1)B_{-i-3}&=0.
\end{aligned}\qquad (39)$$

The complete solutions will thus involve four arbitrary constants; by assigning values to these we may satisfy four equations of condition at the boundary.

This solution covers the case of a spherical shell with bounding surfaces free from stress, or with arbitrary surface stress of a restricted type (corresponding to the restriction imposed on the potential of the bodily forces), or with arbitrary surface displacements of a similarly restricted type.

For the case of the complete sphere, negative powers of x must be rejected, so that the only non-vanishing coefficients in (38) are A_{i-2}, A_i, B_{i-2}, B_i, C_i, D_i; these being connected by four relations included among (39). There remain two arbitrary constants for satisfying the surface conditions (26) and (27).

§ 6. *Case in which the Potential is a Harmonic Function.*

If W is a harmonic function, each term $R_i S_i$ in its development must satisfy Laplace's equation, so that

$$R_i = Ar^i + Br^{-i-1}.$$

Rejecting the negative power, we assume R_i or R to have the form Ar^i. Equations (32) and (33) then reduce to

$$(\lambda + 2\mu)y = -\frac{i}{(2i+1)x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx}\, dx, \tag{40}$$

$$\mu z = -\frac{1}{(2i+1)x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx}\, dx. \tag{41}$$

The further discussion will be restricted to the case in which $i = 2$.

§ 7. *Sphere Strained by Centrifugal Force.*

If the bodily forces are the centrifugal forces due to uniform rotation about a fixed axis, we have

$$W = -\frac{cg}{3a} r^2 P_2, \tag{42}$$

in which c denotes the ratio of equatorial centrifugal force to gravity, and P_2 is the zonal harmonic $\frac{3}{2}\cos^2\theta - \frac{1}{2}$. We now take

$$S_2 = -\tfrac{2}{3} P_2 = \tfrac{1}{3} - \cos^2\theta, \tag{43}$$

$$R_2 = R = \frac{cgr^2}{2a} = \frac{cga}{2} x^2. \tag{44}$$

In accordance with the notation of a former paper,* let e denote the ellipticity of a surface originally spherical and of radius r, and α the angular displacement of a radius vector for which $\theta = 45°$. With the above value of S_2, we have

$$e = \frac{u}{r}, \qquad \alpha = \frac{v}{r}. \tag{45}$$

In writing the complete solutions of the differential equations for this case, we take A_0 and A_2 as arbitrary constants, expressing B_0, B_2, C_2, D_2 in terms of them by means of (39). We thus obtain the equations

$$y = -\frac{7\mu}{\lambda} A_2 x^2 - \frac{cga}{(\lambda + 2\mu)x^3} \int_0^x \rho x^4\, dx, \tag{46}$$

*Transactions American Mathematical Society, vol. 11 (1910), p. 206.

$$z = \frac{7(\lambda+2\mu)}{3\lambda} A_2 x^2 - \frac{cga}{2\mu x^3}\int_0^x \rho x^4\,dx, \tag{47}$$

$$e = A_0 + A_2 x^2 + \frac{cga}{30\mu(\lambda+2\mu)}\left[\frac{5(3\lambda+5\mu)}{x^3}\int_0^x \rho x^4\,dx - \frac{9(\lambda+\mu)}{x^5}\int_0^x \rho x^6\,dx - 2(3\lambda+8\mu)\int_0^x \rho x\,dx\right], \tag{48}$$

$$\alpha = \tfrac{1}{2}A_0 + \frac{5\lambda+7\mu}{6\lambda} A_2 x^2 + \frac{cga}{30\mu(\lambda+2\mu)}\left[\frac{5\mu}{x^3}\int_0^x \rho x^4 dx + \frac{3(\lambda+\mu)}{x^5}\int_0^x \rho x^6 dx - (3\lambda+8\mu)\int_0^x \rho x dx\right]. \tag{49}$$

Substituting these values in (26) and 27), we obtain the following equations for determining A_0 and A_2:

$$2A_0 - A_2 + \frac{cga}{30\mu(\lambda+2\mu)}\left[-10(9\lambda+10\mu)\int_0^1 \rho x^4 dx + 72(\lambda+\mu)\int_0^1 \rho x^6\,dx - 4(3\lambda+8\mu)\int_0^1 \rho x dx\right] = 0, \tag{50}$$

$$A_0 + \frac{8\lambda+7\mu}{3\lambda} A_2 + \frac{cga}{30\mu(\lambda+2\mu)}\left[5(3\lambda+2\mu)\int_0^1 \rho x^4 dx - 24(\lambda+\mu)\int_0^1 \rho x^6\,dx - 2(3\lambda+8\mu)\int_0^1 \rho x dx\right] = 0. \tag{51}$$

Solving, we find

$$A_0 = \frac{cga}{30\mu(\lambda+2\mu)(19\lambda+14\mu)}\left[25(27\lambda^2+56\lambda\mu+28\mu^2)\int_0^1 \rho x^4\,dx - 504(\lambda+\mu)^2\int_0^1 \rho x^6\,dx + 2(3\lambda+8\mu)(19\lambda+14\mu)\int_0^1 \rho x dx\right], \tag{52}$$

$$A_2 = \frac{cga}{30\mu(\lambda+2\mu)(19\lambda+14\mu)}\cdot 360\lambda(\lambda+\mu)\left(\int_0^1 \rho x^6\,dx - \int_0^1 \rho x^4\,dx\right). \tag{53}$$

Of especial interest are the surface values of e and α:

$$e_1 = \frac{cga}{2\mu(19\lambda+14\mu)}\left[5(8\lambda+7\mu)\int_0^1 \rho x^4\,dx - 21(\lambda+\mu)\int_0^1 \rho x^6\,dx\right], \tag{54}$$

$$\alpha_1 = \frac{cga}{4\mu(19\lambda+14\mu)}\left[5\lambda\int_0^1 \rho x^4\,dx + 14(\lambda+\mu)\int_0^1 \rho x^6\,dx\right]. \tag{55}$$

If ρ is constant it is easy to show that the foregoing equations give the following well-known results:

$$e = \frac{c}{2(19\lambda + 14\mu)}\left[2(4\lambda + 3\mu) - (3\lambda + 2\mu)x^2\right]\frac{\rho g a}{\mu}, \tag{56}$$

$$\alpha = \frac{c}{4(19\lambda + 14\mu)}\left[2(4\lambda + 3\mu) - (5\lambda + 4\mu)x^2\right]\frac{\rho g a}{\mu}. \tag{57}$$

§ 8. *Laplace's Law of Density.*

To illustrate the application of the above formulas to a case of variable density, assume Laplace's law, expressed by the formula

$$\rho = \rho_0 \frac{\sin \kappa x}{\kappa x} = \frac{\rho_m \kappa^3}{3(\sin \kappa - \kappa \cos \kappa)} \cdot \frac{\sin \kappa x}{\kappa x}, \tag{58}$$

in which ρ_m is the mean density and ρ_0 the value of ρ at the center. We then find

$$\int_0^x \rho x dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}(1 - \cos \kappa x), \tag{59}$$

$$\frac{1}{x^3}\int_0^x \rho x^4 dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}\left[\frac{6 - \kappa^2 x^2}{\kappa^2 x^2}\cos \kappa x - \frac{6 - \kappa^2 x^2}{\kappa^3 x^3}\sin \kappa x\right], \tag{60}$$

$$\frac{1}{x^5}\int_0^x \rho x^6 dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)}\left[\frac{120 - 60\kappa^2 x^2 + 5\kappa^4 x^4}{\kappa^5 x^5}\sin \kappa x - \frac{120 - 20\kappa^2 x^2 + \kappa^4 x^4}{\kappa^4 x^4}\cos \kappa x\right]. \tag{61}$$

If the ratio of mean density to surface density is assumed to be 2, we find $\kappa = 2.461$, and

$$\int_0^1 \rho x^4 dx = .168\rho_m, \qquad \int_0^1 \rho x^6 dx = .108\rho_m. \tag{62}$$

These values substituted in (54) and (55) give

$$\frac{e_1}{c} = \frac{4.45\lambda + 3.61\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{63}$$

$$\frac{\alpha_1}{c} = \frac{2.35\lambda + 1.51\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}. \tag{64}$$

Comparing with the surface values of e and α for a sphere of uniform density ρ_m,

$$\frac{e_1}{c} = \frac{5\lambda + 4\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{65}$$

$$\frac{\alpha_1}{c} = \frac{3\lambda + 2\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{66}$$

it appears that the assumption of variable density decreases e_1 by about 11 per cent and α_1 by about 22 per cent.

We may also compare the values at the center, which with Laplace's law of density are found to be

$$\frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{A_0}{c} = \frac{124.4\lambda^2 + 349.1\lambda\mu + 191.8\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}, \tag{67}$$

while the corresponding values for uniform density ρ_m are

$$\frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{4\lambda + 3\mu}{19\lambda + 14\mu} \frac{\rho_m g a}{\mu} = \frac{120\lambda^2 + 330\lambda\mu + 180\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}. \tag{68}$$

The central values of e and α are thus slightly increased by the assumption of variable density.

§ 9. *Gravitating Sphere.*

The results of the above solution cannot be applied directly to a gravitating body of size and mass comparable with those of the earth, since the changes in the gravitational forces caused by the strain will be of the same order of importance as the disturbing forces. There is no difficulty in expressing the gravitational forces in terms of the strain and the density and thus forming the differential equations which must replace (16) and (17) above given, but the solution of these equations appears to involve serious analytical difficulties.*

* In expressing the equations of interior equilibrium in terms of the strain, for the case of a gravitating body, it is necessary to assume that the changes of stress and strain from their initial values are related in the same way as if the initial condition were one of zero stress and strain. This assumption is the basis of the solution given for the case of a compressible sphere in the paper cited above, the reasoning being in this respect substantially identical with that previously applied to the case of incompressibility. (See A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 2d ed., p. 249.) It has sometimes been assumed that this reasoning is valid only in the case of incompressibility. Thus LOVE states that if the sphere is homogeneous and incompressible, the initial stress may be taken to be of the nature of a hydrostatic pressure, its value being then easily determined, and the terms which correspond to this initial stress in the equations of equilibrium being found to cancel with those representing the initial body-force. It seems clear, however, first that even in an incompressible body the initial condition may involve tangential stress, and secondly that the assumption of no initial tangential stress is not essential to the argument. Whatever values the initial stress-components may have, the terms representing them in the equations of equilibrium will cancel the terms representing the initial body-forces.

If the body is assumed incompressible and of uniform density, the effect of gravitation can be taken account of synthetically by a method employed by Kelvin.* This method consists in combining the results for the two cases of a gravitating fluid and a non-gravitating elastic solid. It does not appear that similar reasoning can be applied if either of the assumptions of incompressibility and uniform density is dropped, since the distribution of the forces of gravitation cannot be assumed to be the same for the strained elastic solid as for the fluid.

The method of applying this principle in the solutions cited above, both for the case of incompressibility and for that of compressibility, consists in comparing the forces acting upon a volume-element in the strained condition with those initially acting upon the same volume-element. It would perhaps be more reasonable to deal with an individual element of matter, using the increments of the stress-components acting on that element in expressing the stress-strain relations by the usual formulas. The increment of stress for an individual element would be in equilibrium with the increment of body-force acting upon that element, which would consist in part of the change in the gravitational force due to the displacement of the attracted element. This would introduce additional terms into the equations of equilibrium, but would add no difficulty to their solution.

* THOMSON and TAIT's *Natural Philosophy*, § 840, Ed. of 1890.